Semantics and Verification of Software

Lecture 9: Axiomatic Semantics of WHILE II (Hoare Logic)

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Outline

1. Recapitulation: Axiomatic Semantics of WHILE
2. Proof Rules for Partial Correctness
3. Soundness of Hoare Logic

Partial Correctness Properties

Validity of property $\{A\}\, c\, \{B\}$

For all states $\sigma \in \Sigma$ which satisfy A:
if the execution of c in σ terminates in $\sigma' \in \Sigma$, then σ' satisfies B.

Syntax of Assertion Language

Definition (Syntax of assertions)

The syntax of *Assn* is defined by the following context-free grammar:

$$a ::= z \mid x \mid i \mid a_1{+}a_2 \mid a_1{-}a_2 \mid a_1{*}a_2 \in LExp$$
$$A ::= t \mid a_1{=}a_2 \mid a_1{>}a_2 \mid \neg A \mid A_1 \wedge A_2 \mid A_1 \vee A_2 \mid \forall i.A \in Assn$$

- Thus: $AExp \subsetneq LExp$, $BExp \subsetneq Assn$
- The following (and other) abbreviations will be employed:

$$A_1 \Rightarrow A_2 := \neg A_1 \vee A_2$$
$$\exists i.A := \neg(\forall i.\neg A)$$
$$a_1 \geq a_2 := a_1{>}a_2 \vee a_1{=}a_2$$
$$\vdots$$

The semantics now additionally depends on values of logical variables:

Definition (Semantics of *LExp*)

An interpretation is an element of the set $Int := \{I \mid I : LVar \to \mathbb{Z}\}$. The value of an arithmetic expressions with logical variables is given by the functional

$$\mathfrak{L}[\![.]\!] : LExp \to (Int \to (\Sigma \to \mathbb{Z}))$$

where

$$\begin{array}{rclrcl}
\mathfrak{L}[\![z]\!]I\sigma & := & z & \mathfrak{L}[\![a_1 + a_2]\!]I\sigma & := & \mathfrak{L}[\![a_1]\!]I\sigma + \mathfrak{L}[\![a_2]\!]I\sigma \\
\mathfrak{L}[\![x]\!]I\sigma & := & \sigma(x) & \mathfrak{L}[\![a_1 - a_2]\!]I\sigma & := & \mathfrak{L}[\![a_1]\!]I\sigma - \mathfrak{L}[\![a_2]\!]I\sigma \\
\mathfrak{L}[\![i]\!]I\sigma & := & I(i) & \mathfrak{L}[\![a_1 * a_2]\!]I\sigma & := & \mathfrak{L}[\![a_1]\!]I\sigma \cdot \mathfrak{L}[\![a_2]\!]I\sigma
\end{array}$$

Semantics of Assertions II

Reminder: $A ::= t \mid a_1{=}a_2 \mid a_1{>}a_2 \mid \neg A \mid A_1 \wedge A_2 \mid A_1 \vee A_2 \mid \forall i.A \in Assn$

Definition (Semantics of assertions)

Let $A \in Assn$, $\sigma \in \Sigma_\perp$, and $I \in Int$. The relation "σ satisfies A in I" (notation: $\sigma \models^I A$) is inductively defined by:

$$
\begin{array}{lll}
\sigma \models^I \text{true} & & \\
\sigma \models^I a_1{=}a_2 & \text{if } \mathfrak{L}[\![a_1]\!]I\sigma = \mathfrak{L}[\![a_2]\!]I\sigma \\
\sigma \models^I a_1{>}a_2 & \text{if } \mathfrak{L}[\![a_1]\!]I\sigma > \mathfrak{L}[\![a_2]\!]I\sigma \\
\sigma \models^I \neg A & \text{if not } \sigma \models^I A \\
\sigma \models^I A_1 \wedge A_2 & \text{if } \sigma \models^I A_1 \text{ and } \sigma \models^I A_2 \\
\sigma \models^I A_1 \vee A_2 & \text{if } \sigma \models^I A_1 \text{ or } \sigma \models^I A_2 \\
\sigma \models^I \forall i.A & \text{if } \sigma \models^{I[i \mapsto z]} A \text{ for every } z \in \mathbb{Z} \\
\perp \models^I A & &
\end{array}
$$

Furthermore σ satisfies A ($\sigma \models A$) if $\sigma \models^I A$ for every interpretation $I \in Int$, and A is called valid ($\models A$) if $\sigma \models A$ for every state $\sigma \in \Sigma$.

Partial Correctness Properties

Definition (Partial correctness properties)

Let $A, B \in \mathit{Assn}$ and $c \in \mathit{Cmd}$.

- An expression of the form $\{A\}\,c\,\{B\}$ is called a partial correctness property with precondition A and postcondition B.
- Given $\sigma \in \Sigma_\perp$ and $I \in \mathit{Int}$, we let
$$\sigma \models^I \{A\}\,c\,\{B\}$$
if $\sigma \models^I A$ implies $\mathfrak{C}[\![c]\!]\sigma \models^I B$
(or equivalently: $\sigma \in A^I \Rightarrow \mathfrak{C}[\![c]\!]\sigma \in B^I$).
- $\{A\}\,c\,\{B\}$ is called valid in I (notation: $\models^I \{A\}\,c\,\{B\}$) if $\sigma \models^I \{A\}\,c\,\{B\}$ for every $\sigma \in \Sigma_\perp$ (or equivalently: $\mathfrak{C}[\![c]\!]A^I \subseteq B^I$).
- $\{A\}\,c\,\{B\}$ is called valid (notation: $\models \{A\}\,c\,\{B\}$) if $\models^I \{A\}\,c\,\{B\}$ for every $I \in \mathit{Int}$.

Outline

Hoare Logic I

Goal: syntactic derivation of valid partial correctness properties. Here $A[x \mapsto a]$ denotes the syntactic replacement of every occurrence of x by a in A.

Tony Hoare (* 1934)

Definition 9.1 (Hoare Logic)

The Hoare rules are given by

$$\text{(skip)}\ \frac{}{\{A\}\ \texttt{skip}\ \{A\}} \qquad \text{(asgn)}\ \frac{}{\{A[x \mapsto a]\}\ x\texttt{:=}a\ \{A\}}$$

$$\text{(seq)}\ \frac{\{A\}\ c_1\ \{C\} \quad \{C\}\ c_2\ \{B\}}{\{A\}\ c_1\,;c_2\ \{B\}} \qquad \text{(if)}\ \frac{\{A \wedge b\}\ c_1\ \{B\} \quad \{A \wedge \neg b\}\ c_2\ \{B\}}{\{A\}\ \texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2\ \{B\}}$$

$$\text{(while)}\ \frac{\{A \wedge b\}\ c\ \{A\}}{\{A\}\ \texttt{while}\ b\ \texttt{do}\ c\ \{A \wedge \neg b\}}$$

$$\text{(cons)}\ \frac{\models (A \Rightarrow A') \quad \{A'\}\ c\ \{B'\} \quad \models (B' \Rightarrow B)}{\{A\}\ c\ \{B\}}$$

A partial correctness property is provable (notation: $\vdash \{A\}\ c\ \{B\}$) if it is derivable by the Hoare rules. In (while), A is called a (loop) invariant.

Example 9.2 (Factorial program)

Proof of $\{A\}$ `y:=1;`c $\{B\}$ where

$$c := (\texttt{while}\ \neg(\texttt{x=1})\ \texttt{do}\ (\texttt{y:=y*x;}\ \texttt{x:=x-1}))$$
$$A := (\mathtt{x} > 0 \wedge \mathtt{x} = i)$$
$$B := (\mathtt{y} = i!)$$

(on the board)

Example 9.2 (Factorial program)

Proof of $\{A\}\, \texttt{y:=1;}c\, \{B\}$ where

$$\begin{aligned} c &:= (\texttt{while } \neg\texttt{(x=1) do (y:=y*x; x:=x-1)}) \\ A &:= (\texttt{x} > 0 \wedge \texttt{x} = i) \\ B &:= (\texttt{y} = i!) \end{aligned}$$

(on the board)

Structure of the proof:

$$(\text{seq})\,\dfrac{(\text{cons})\,\dfrac{\overline{4}\quad (\text{asgn})\,\overline{5}\quad \overline{6}}{2}\qquad (\text{cons})\,\dfrac{\overline{7}\quad (\text{while})\,\dfrac{(\text{cons})\,\dfrac{\overline{11}\quad (\text{seq})\,\dfrac{(\text{asgn})\,\overline{14}\quad (\text{asgn})\,\overline{15}}{12}\quad \overline{13}}{10}}{8}\quad \overline{9}}{3}}{1}$$

Hoare Logic III

Example 9.2 (continued)

Here the respective propositions are given by (where $C := (\mathrm{x} > 0 \wedge \mathrm{y} * \mathrm{x}! = i!)$):

1. $\{A\}\,\texttt{y := 1;}c\,\{B\}$
2. $\{A\}\,\texttt{y := 1}\,\{C\}$
3. $\{C\}\,c\,\{B\}$
4. $\models (A \Rightarrow C[\mathrm{y} \mapsto 1])$
5. $\{C[\mathrm{y} \mapsto 1]\}\,\texttt{y := 1}\,\{C\}$
6. $\models (C \Rightarrow C)$
7. $\models (C \Rightarrow C)$
8. $\{C\}\,c\,\{\neg(\neg(\texttt{x = 1})) \wedge C\}$
9. $\models (\neg(\neg(\texttt{x = 1})) \wedge C \Rightarrow B)$
10. $\{\neg(\texttt{x = 1}) \wedge C\}\,\texttt{y := y*x; x := x-1}\,\{C\}$
11. $\models (\neg(\texttt{x = 1}) \wedge C \Rightarrow C[\mathrm{x} \mapsto \texttt{x-1}, \mathrm{y} \mapsto \texttt{y*x}])$
12. $\{C[\mathrm{x} \mapsto \texttt{x-1}, \mathrm{y} \mapsto \texttt{y*x}]\}\,\texttt{y := y*x; x := x-1}\,\{C\}$
13. $\models (C \Rightarrow C)$
14. $\{C[\mathrm{x} \mapsto \texttt{x-1}, \mathrm{y} \mapsto \texttt{y*x}]\}\,\texttt{y := y*x}\,\{C[\mathrm{x} \mapsto \texttt{x-1}]\}$
15. $\{C[\mathrm{x} \mapsto \texttt{x-1}]\}\,\texttt{x := x-1}\,\{C\}$

Outline

Soundness of Hoare Logic I

Soundness: no wrong propositions can be derived, i.e., every (syntactically) provable partial correctness property is also (semantically) valid

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For the corresponding proof we use:

Lemma 9.3 (Substitution lemma)

For every $A \in Assn$, $x \in Var$, $a \in AExp$, $\sigma \in \Sigma$, *and* $I \in Int$:

$$\sigma \models^I A[x \mapsto a] \iff \sigma[x \mapsto \mathfrak{A}[\![a]\!]\sigma] \models^I A.$$

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$$\sigma \models^I A[x \mapsto a] \iff \sigma[x \mapsto \mathfrak{A}[\![a]\!]\sigma] \models^I A.$$

Proof.

by induction over $A \in Assn$ (omitted) □

Soundness of Hoare Logic II

Theorem 9.4 (Soundness of Hoare Logic)

For every partial correctness property $\{A\}\,c\,\{B\}$,

$$\vdash \{A\}\,c\,\{B\} \quad \Rightarrow \quad \models \{A\}\,c\,\{B\}.$$

Soundness of Hoare Logic II

Theorem 9.4 (Soundness of Hoare Logic)

For every partial correctness property $\{A\}\, c\, \{B\}$,

$$\vdash \{A\}\, c\, \{B\} \quad \Rightarrow \quad \models \{A\}\, c\, \{B\}.$$

Proof.

Let $\vdash \{A\}\, c\, \{B\}$. By induction over the structure of the corresponding proof tree we show that, for every $\sigma \in \Sigma$ and $I \in Int$ such that $\sigma \models^I A$, $\mathfrak{C}[\![c]\!]\sigma \models^I B$ (on the board).
(If $\sigma = \bot$, then $\mathfrak{C}[\![c]\!]\sigma = \bot \models^I B$ holds trivially.) □