From Measure Theory to CTMDPs

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Measure Theory

Our Setting

Assume a set Ω, called **sample space**.
Subsets A of Ω are called **events**.
Idea: Measure the {size | probability | volume | length} of events!

Intuition: Let $\omega \in \Omega$ be the outcome of an experiment.
Then A is an event if $\omega \in A$ can be decided.

Fields and σ–fields

Definition (Field)

A class of subsets $\mathfrak{F}$ of Ω is a field iff

1. $\Omega \in \mathfrak{F}$.
2. $A \in \mathfrak{F} \Rightarrow A^c \in \mathfrak{F}$.
3. $A_1, \ldots, A_n \in \mathfrak{F} \implies \bigcup_{i=1}^{n} A_i \in \mathfrak{F}$
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Let $\mathcal{C} \subseteq 2^{\Omega}$. $\sigma(\mathcal{C})$ denotes the **smallest σ–field** containing $\mathcal{C}$.

Example: The Borel σ–field

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Define the class of **finite disjoint unions** of right–semiclosed intervals:

$$\mathfrak{F}_0(\mathbb{R}) := \{I_1 \uplus I_2 \uplus \cdots \uplus I_n \mid n \in \mathbb{N}\}.$$

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$\mathfrak{F}_0(\mathbb{R})$ is a field.

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Let $\mathcal{E} = \{(a, b] \mid a, b \in \mathbb{R} \text{ and } a < b\}$ and
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Example

$\mathfrak{B}(\mathbb{R})$ has many generators:

- $\mathfrak{F}_0(\mathbb{R})$, the set of finite disjoint unions of right–semiclosed intervals,
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Intuition: Construct σ–field by forming countable unions and complements of intervals in all possible ways.

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Let $\mathfrak{F}$ be a σ–field over subsets of Ω. A **measure** is a function

$$\mu : \mathfrak{F} \to \bar{\mathbb{R}}_{\geqslant 0} \qquad \text{where } \bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$$

which is **countably additive**:

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) \qquad \text{for disjoint sets } A_i \in \mathfrak{F}.$$

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Remark: If $\mu(\Omega) = 1$, μ is a **probability measure**.

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Let $I_1 \uplus \cdots \uplus I_n \in \mathfrak{F}_0(\mathbb{R})$. Extend μ to $\mathfrak{F}_0(\mathbb{R})$ by defining

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But: What about $\mu(A)$ for **arbitrary** $A \in \mathfrak{B}(\mathbb{R})$?

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Theorem (Carathéodory Extension Theorem)

Let $\mathfrak{F}_0$ be a field over subsets of a set Ω and let μ be a **measure** *on $\mathfrak{F}_0$. If μ is σ–finite, i.e.*

$$\Omega = \bigcup_{i=1}^{\infty} A_i \qquad \text{where } A_i \in \mathfrak{F}_0 \text{ and } \mu(A_i) < \infty,$$

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In practice: Avoid the σ–field whenever possible!

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where $A_1, A_2, \dots \in \mathfrak{F}_0(\mathbb{R})$, $\bigcup_{j=1}^{\infty} A_j \in \mathfrak{F}_0$ and the A_j disjoint.

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Theorem

Let $F : \mathbb{R} \to \mathbb{R}$ be a distrib. function. Let $\mu(a,b] := F(b) - F(a)$. There is a unique extension of μ to a Lebesgue–Stieltjes measure on $\mathbb{R}$.

Thus: Countable additivity of μ follows by defining $F(x) := x$.

Lebesgue's Intuition

Lebesgue about his integral

"One might say that Riemann's approach is comparable to a messy merchant who counts coins in the order they come to his hand whereas we act like a prudent merchant who says:

- I have A_1 coins à one crown, that is $A_1 \cdot 1$ crowns,
- A_2 coins à two crowns, that is $A_2 \cdot 2$ crowns and
- A_3 coins à five crowns, that is $A_3 \cdot 5$ crowns.

Therefore I have $A_1 \cdot 1 + A_2 \cdot 2 + A_3 \cdot 5$ crowns.

Both approaches – no matter how rich the merchant might be – lead to the same result since he only has to count a finite number of coins.
But for us who must add infinitly many indivisibles, the difference between the approaches is essential."

H.Lebesgue, 1926

Measurability

Definition (Measurability)

Let Ω_1, Ω_2 be sets with associated σ–fields $\mathfrak{F}_1$ and $\mathfrak{F}_2$.
$h : \Omega_1 \to \Omega_2$ is **measurable** iff

$$h^{-1}(A) \in \mathfrak{F}_1 \qquad \text{for each } A \in \mathfrak{F}_2$$

Notation: $h : (\Omega_1, \mathfrak{F}_1) \to (\Omega_2, \mathfrak{F}_2)$.

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Some remarks:

- h is **Borel measurable** if $h : (\Omega, \mathfrak{F}) \to (\bar{\mathbb{R}}, \mathfrak{B}(\bar{\mathbb{R}}))$.

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- In probability theory, h is called a **random variable**.

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If h is a **simple** function, it can be represented as

$$h(\omega) := \sum_{i=1}^{n} x_i \cdot \mathbf{I}_{A_i}(\omega)$$

where $A_i \in \mathfrak{F}$ are pairwise disjoint.

$\mathbf{I}_{A_i}$ denotes the indicator function $\mathbf{I}_{A_i}(\omega) := \begin{cases} 1 & \text{if } \omega \in A_i \\ 0 & \text{otherwise} \end{cases}$.

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Intuition: Choose A_i as the **preimage** of x_i!

Lebesgue Integral

Definition (Lebesgue Integral for **Simple** Functions)

Let $(\Omega, \mathfrak{F}, \mu)$ be a measure space, $h : \Omega \to \bar{\mathbb{R}}$ simple:

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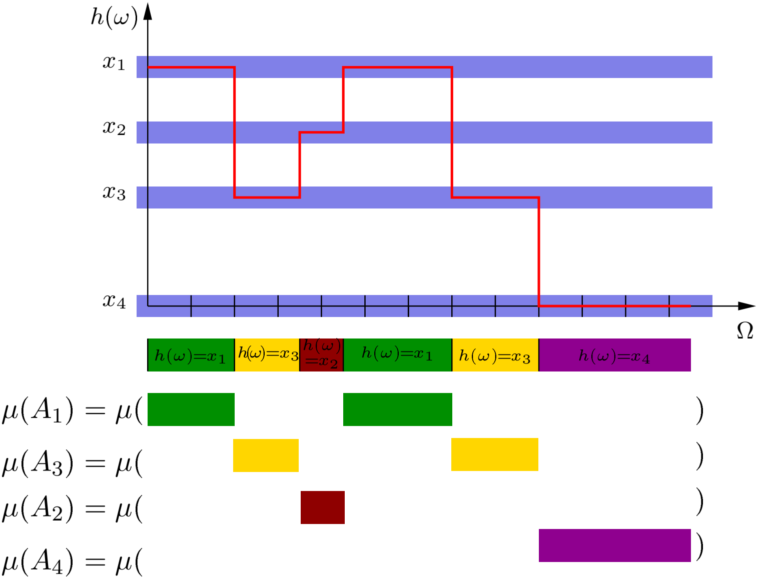
The **Lebesgue–integral** of h is defined as

$$\int_\Omega h \, d\mu := \sum_{i=1}^{n} x_i \cdot \mu(A_i).$$

Intuition: Multiply each x_i with the measure of its preimage A_i.

Example: Lebesgue Integral



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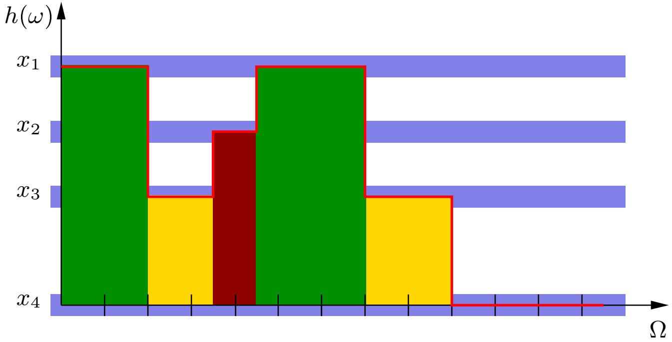
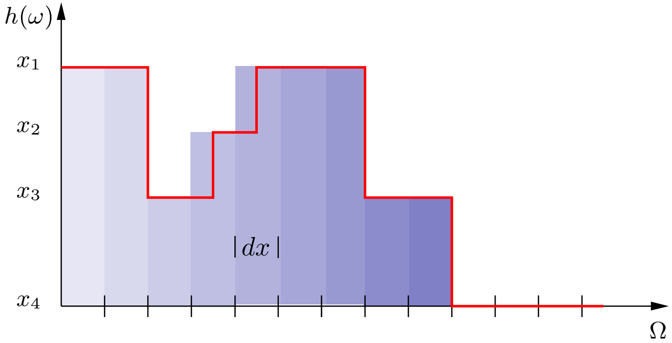


$$\int_\Omega h \; d\mu = x_1\mu(A_1) + x_2\mu(A_2) + x_3\mu(A_3)$$

Example: Riemann (Darboux) Integral



Lebesgue Integral on Nonnegative Functions

Definition

If h is **nonnegative Borel measurable**, then

$$\int_\Omega h \, d\mu := \sup \left\{ \int_\Omega s \, d\mu \mid s \text{ is simple and } 0 \leqslant s \leqslant h \right\}.$$

Theorem

A nonnegative Borel measurable function h is the limit of an increasing sequence of nonnegative simple functions h_n.

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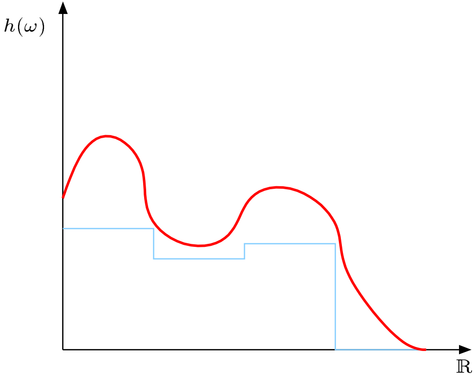

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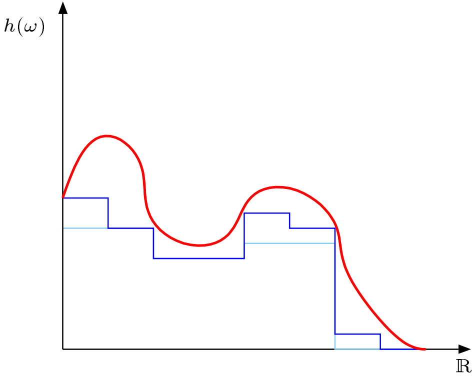

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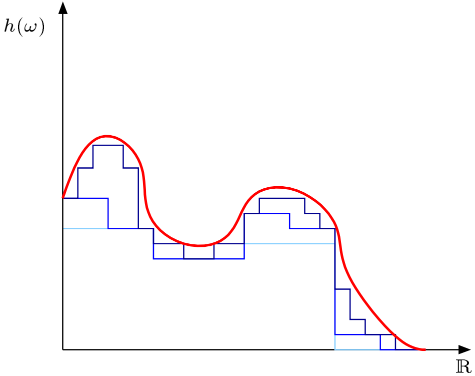

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- $\Omega = \Omega_1 \times \cdots \times \Omega_n$
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- The **product σ–field** $\mathfrak{F}$ is the smallest σ–field containing all measurable rectangles:

$$\mathfrak{F} := \sigma\Big(\mathfrak{F}_1 \times \mathfrak{F}_2 \times \cdots \times \mathfrak{F}_n\Big)$$

Measures on Finite Product Spaces

To start with: Only products of **two σ–fields**!

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To start with: Only products of **two σ–fields**!

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3. uniformly σ–finite:
   $\Omega_2 = \bigcup_{n=1}^{\infty} B_n$ where $\mu(\omega_1, B_n) \leqslant k_n$ for all ω_1 and fixed $k_n \in \mathbb{R}$.

Measures on Finite Product Spaces

Theorem (Product Measure Theorem)

Given $(\Omega_1, \mathfrak{F}_1, \mu_1)$, $(\Omega_2, \mathfrak{F}_2)$ *and* $\mu(\omega_1, \cdot)$ *as before.*

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μ *is defined (now on the entire* σ*–field) as follows:*

$$\mu(F) := \int_{\Omega_1} \mu(\omega_1, F(\omega_1))\ \mu_1(d\omega_1), \qquad \text{for all } F \in \mathfrak{F}$$

where $F(\omega_1) := \{\omega_2 \mid (\omega_1, \omega_2) \in F\}$.

Lebesgue Integrals on Finite Product Spaces

Theorem (Fubini's Theorem)

Let $f : (\Omega, \mathfrak{F}) \to (\bar{\mathbb{R}}, \mathfrak{B}(\bar{\mathbb{R}}))$*. If* f *is* **nonnegative***, then*

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$$\int_{\Omega} f\ d\mu = \int_{\Omega_1} \Big(\int_{\Omega_2} f(\omega_1, \omega_2)\ \mu(\omega_1, d\omega_2) \Big) \mu_1(d\omega_1).$$

Justification of iterated integration!

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2. is measurable, i.e. for all fixed $C \in \mathfrak{F}_{j+1}$:

$$\mu(\omega_1, \ldots, \omega_j, C) : \big(\Omega_1 \times \cdots \times \Omega_j, \sigma(\mathfrak{F}_1 \times \cdots \times \mathfrak{F}_j)\big) \to \big(\bar{\mathbb{R}}, \mathfrak{B}(\bar{\mathbb{R}})\big)$$

3. uniformly σ–finite.

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Let $f : (\Omega, \mathfrak{F}) \to (\bar{\mathbb{R}}, \mathfrak{B}(\bar{\mathbb{R}}))$*. If* $f \geqslant 0$*, then*

$$\int_{\Omega} f \, d\mu = \int_{\Omega_1} \mu_1(d\omega_1) \int_{\Omega_2} \mu(\omega_1, d\omega_2) \cdots \int_{\Omega_n} f(\omega_1, \ldots, \omega_n) \, \mu(\omega_1, \ldots, \omega_{n-1}, d\omega_n).$$

Measures on Infinite Product Spaces

Definition (Cylinder Set)

Let $(\Omega_j, \mathfrak{F}_j)$ be a measurable space, $j = 1, 2, \ldots$.
Let $\Omega = \times_{j=1}^{\infty} \Omega_j$. If $B^n \subseteq \Omega_1 \times \cdots \times \Omega_n$, define

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- B_n is **measurable** if $B^n \in \sigma(\mathfrak{F}_1 \times \cdots \times \mathfrak{F}_n)$.
- B_n is a **rectangle** if $B^n = A_1 \times \cdots \times A_n$ and $A_j \subseteq \Omega_j$;
  B_n is a **measurable rectangle** if $A_j \in \mathfrak{F}_j$.

Measures on Infinite Product Spaces

Ionescu–Tulcea Theorem

Let P_1 be a **probability measure** on $\mathfrak{F}_1$ and **for each** $(\omega_1, \ldots, \omega_j)$, $j \in \mathbb{N}$, assume a measurable probability measure $P(\omega_1, \ldots, \omega_j, \cdot)$ on $\mathfrak{F}_{j+1}$.

Let P_n be defined on $\sigma(\mathfrak{F}_1 \times \cdots \times \mathfrak{F}_n)$:

$$P_n(F) = \int_{\Omega_1} P_1(d\omega_1) \int_{\Omega_2} P(\omega_1, d\omega_2) \cdots \int_{\Omega_n} \mathbf{I}_F(\omega_1, \ldots, \omega_n)\, P(\omega_1, \ldots, \omega_{n-1}, d\omega_n).$$

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Intuition: The measure of a cylinder equals the measure of its finite base.