Satisfiability Checking

First Order Logic

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Theory of Hybrid Systems
Informatik 2

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First-Order Logic

- First-order (FO) logic is a framework.
- It gives us a generic syntax by recursively defining strings on an alphabet.
- Non-logical elements are logically combined using:
  - constants (Eve, 0, true, . . .)
  - variables (x,y,. . .)
  - function symbols ($+(\cdot,\cdot)$, vaterOf$(\cdot)$,. . .)
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  - logical symbols $((,),\wedge,\neg,\ldots,\exists,\forall)$.

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  - logical symbols $((,), \wedge, \neg, \ldots, \exists, \forall)$.

Note:

- Constants can also be seen as function symbols of arity 0.
- Sometimes equality $(=)$ is included as a logical symbol.
- E.g., the Boolean connectives negation $(\neg)$ and conjunction $(\wedge)$ and the existential quantifier $\exists$ would be sufficient, the remaining syntax $(\vee, \rightarrow, \leftrightarrow, \ldots, \forall)$ are syntactic sugar.

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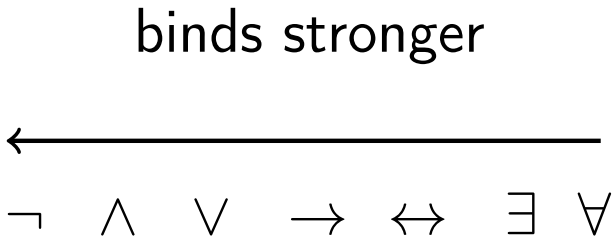
The formulae obtained by the first rule are said to be atomic.

Notational conventions

We omit parenthesis whenever we may restore them through operator precedence:



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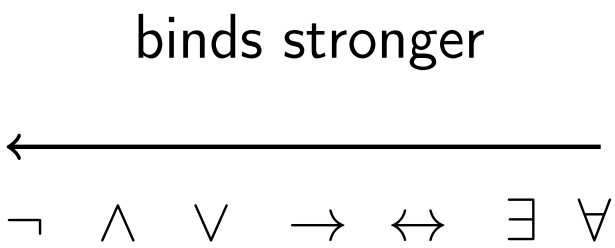

- Thus, we write:

  $\neg\neg a$ for $(\neg(\neg a))$,

  $\exists a.\ \exists b.\ (a \wedge b \rightarrow F(a,b))$ for $\exists a.\ \exists b.\ ((a \wedge b) \rightarrow F(a,b))$

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- The same rule holds with $\forall$ in place of $\exists$.

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- In $z \vee \forall x.\ \forall y.\ (P(x) \rightarrow Q(z))$, x and y are bound variables, z is a free variable, and w is neither bound nor free.
- In $z \vee \forall z.P(z)$, z is both bound and free.

Freeness and boundness can be also specialized to specific occurrences of variables in a formula. In $z \vee \forall z.P(z)$, the first occurrence of z is free while the second is bound.

Some definitions

- A signature Σ fixes a set of non-logical symbols.
- A Σ-formula is a formula with non-logical symbols from Σ.
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In the previous example:
$\Sigma = (\mathit{Eve}, \mathit{isWoman}(\cdot), \mathit{lovesShopping}(\cdot))$ with

- *Eve* a constant and
- *isWoman* and *lovesShopping* unary predicate symbols.

The formulae

1. $\forall x.\ \mathit{isWoman}(x) \rightarrow \mathit{lovesShopping}(x)$
2. *isWoman(Eve)*
3. *lovesShopping(Eve)*

are Σ-sentences (the only variable x is bound).

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- Examples of Σ-formulae:

$$\exists x.\ \forall y.\ x > y$$
$$\forall x.\ \exists y.\ x > y$$
$$\forall x.\ x + 1 > x$$
$$\forall x.\ \neg(x + 0 > x \vee x > x + 0)$$

Examples

- $\Sigma = \{0, 1, +, *, <, \mathit{isPrime}\}$
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- An example Σ-formula:

$$\forall n.\ \exists p.\ 1 < n \rightarrow \mathit{isPrime}(p) \wedge n < p < 2 * n$$

Example

- Let $\Sigma = \{0, 1, +, =\}$ where $0, 1$ are constants, $+$ is a binary function symbol and $=$ a binary predicate symbol.
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- Q: Is φ true in $\mathbb{N}_0$?
- A: Depends on the interpretation of '$+$' and '$=$'!

Structures, satisfiability, validity

- A structure is given by:
  - a domain D,
  - an interpretation I of the non-logical symbols that
    - maps each constant symbol to a domain element,
    - maps each function symbol to a function of the same arity, and
    - maps each predicate symbol to a predicate of the same arity,
  - an assignment of a domain element to each free (unquantified) variable.

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- A formula is valid if it is satisfied by all structures.

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constants: $[\![c]\!]_{(D,I)} = I(c)$

variables: $[\![x]\!]_{(D,I)} = I(x)$

functions: $[\![f(t_1, \ldots, t_n)]\!]_{(D,I)} = I(f)([\![t_1]\!]_{(D,I)}, \ldots, [\![t_n]\!]_{(D,I)})$

predicates: $[\![p(t_1, \ldots, t_n)]\!]_{(D,I)} = I(p)([\![t_1]\!]_{(D,I)}, \ldots, [\![t_n]\!]_{(D,I)})$

logical structure:

$$[\![\neg\varphi]\!]_{(D,I)} = \begin{cases} \mathit{true} & \mathit{if}\ [\![\varphi]\!]_{(D,I)} = \mathit{false} \\ \mathit{false} & \mathit{if}\ [\![\varphi]\!]_{(D,I)} = \mathit{true} \end{cases}$$

$$[\![\varphi \wedge \psi]\!]_{(D,I)} = \begin{cases} \mathit{true} & \mathit{if}\ [\![\varphi]\!]_{(D,I)} = \mathit{true}\ \mathit{and}\ [\![\psi]\!]_{(D,I)} = \mathit{true} \\ \mathit{false} & \mathit{if}\ [\![\varphi]\!]_{(D,I)} = \mathit{false}\ \mathit{or}\ [\![\psi]\!]_{(D,I)} = \mathit{false} \end{cases}$$

$$[\![\exists x.\ \varphi]\!]_{(D,I)} = \begin{cases} \mathit{true} & \mathit{if\ there\ exists}\ v \in D\ \mathit{such\ that}\ [\![\varphi]\!]_{(D,I[x \mapsto v])} = \mathit{true} \\ \mathit{false} & \mathit{if\ for\ all}\ v \in D\ \mathit{we\ have}\ [\![\varphi]\!]_{(D,I[x \mapsto v])} = \mathit{false} \end{cases}$$

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- Q: Is φ satisfiable?
- A: Yes. Consider the structure S:
  - Domain: $\mathbb{N}_0$
  - Interpretation:
    - 0 and 1 are mapped to 0 and 1 in $\mathbb{N}_0$
    - $=$ means equality
    - $+$ means addition

  S satisfies φ. S is said to be a model of φ.

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- Q: Is φ valid?
- A: No. Consider the structure S':
  - Domain: $\mathbb{N}_0$
  - Interpretation:
    - 0 and 1 are mapped to 0 and 1 in $\mathbb{N}_0$
    - $=$ means equality
    - $+$ means multiplication

  S' does not satisfy φ.

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- A Σ-formula φ is T-satisfiable if there exists a structure that satisfies both the sentences of T and φ.
- A Σ-formula φ is T-valid if all structures that satisfy the sentences defining T also satisfy φ.

Examples

- $\Sigma = \{0, 1, +, =\}$
- $\varphi = \exists x.\ x + 0 = 1$ a Σ-formula.
- We now define the Σ-theory T by the following axioms:
  1. $\forall x.\ x = x \quad //=$ must be reflexive
  2. $\forall x.\ \forall y.\ x + y = y + x \quad //+$ must be commutative

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- A: No. S' satisfies the sentences in T but not φ.

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- A: Yes. (S' does not satisfy the third axiom.)

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- We now define the Σ-theory T by the following axioms:
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- Q: Is φ T-valid?
- A: Yes. Every structure that satisfies T also satisfies φ.

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- $\varphi : \forall x.\ \exists y.\ y < x$ a Σ-formula
- Consider the Σ-theory T defined by the axioms:
  1. $\forall x.\ \forall y.\ \forall z.\ x < y \wedge y < z \rightarrow x < z$ (transitivity)
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- A: No. We construct a structure to the contrary:
  - Domain: $\mathbb{N}_0$
  - $<$ means "less than"

Logic fragments

- So far we only restricted the non-logical symbols by signatures and their interpretation by theories.
- Sometimes we want to restrict the grammar and the logical symbols that we can use as well.
- These are called logic fragments.
- Examples:
  - The quantifier-free fragment over $\Sigma = \{0, 1, =, +\}$
  - The conjunctive fragment over $\Sigma = \{0, 1, =, +\}$

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  (A very degenerate one.)

- Q: What is T?
- A: Quantified Boolean formulae (QBF)

  Example:
  - $\forall x_1.\ \exists x_2.\ \forall x_3.\ x_1 \rightarrow (x_2 \vee x_3)$

Some famous theories

- Presburger arithmetic: $\Sigma = \{0, 1, +, =\}$ over integers
- Peano arithmetic: $\Sigma = \{0, 1, +, *, =\}$ over integers
- Linear real argebra: $\Sigma = \{0, 1, +, =\}$ over reals
- Real algebra: $\Sigma = \{0, 1, +, *, =\}$ over reals
- Theory of arrays
- Theory of pointers
- ...

The algorithmic point of view...

- It is also common to present theories NOT through the axioms that define them.
- The interpretation of symbols is fixed to their common use.
  - Thus $+$ is plus, ...
- The fragment is defined via grammar rules rather than restrictions on the generic first-order grammar.

The algorithmic point of view...

- Example: Equality logic
- Grammar:

$$\begin{aligned} \textit{formula} ::= &\quad \textit{atom} \mid \textit{formula} \wedge \textit{formula} \mid \neg \textit{formula} \\ \\ \textit{atom} ::= &\quad \textit{Boolean-variable} \mid \\ &\quad \textit{variable} = \textit{variable} \mid \\ &\quad \textit{variable} = \textit{constant} \mid \\ &\quad \textit{constant} = \textit{constant} \end{aligned}$$

- Interpretation: $=$ is equality.

Expressiveness of a theory

- Each formula defines a language:
  The set of satisfying assignments (models) are the words accepted by this language.

- Consider the fragment '2-CNF':

$$\begin{array}{lcl} \textit{formula} & ::= & (\textit{literal} \vee \textit{literal}) \mid \textit{formula} \wedge \textit{formula} \\ \textit{literal} & ::= & \textit{Boolean-variable} \mid \neg \textit{Boolean-variable} \end{array}$$

- Example formula:

$$(x_1 \vee \neg x_2) \wedge (\neg x_3 \vee x_2)$$

Expressiveness of a theory

- Now consider the propositional logic formula $\varphi : (x_1 \vee x_2 \vee x_3)$
- Q: Can we express this language with 2-CNF?

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- Proof:

Expressiveness of a theory

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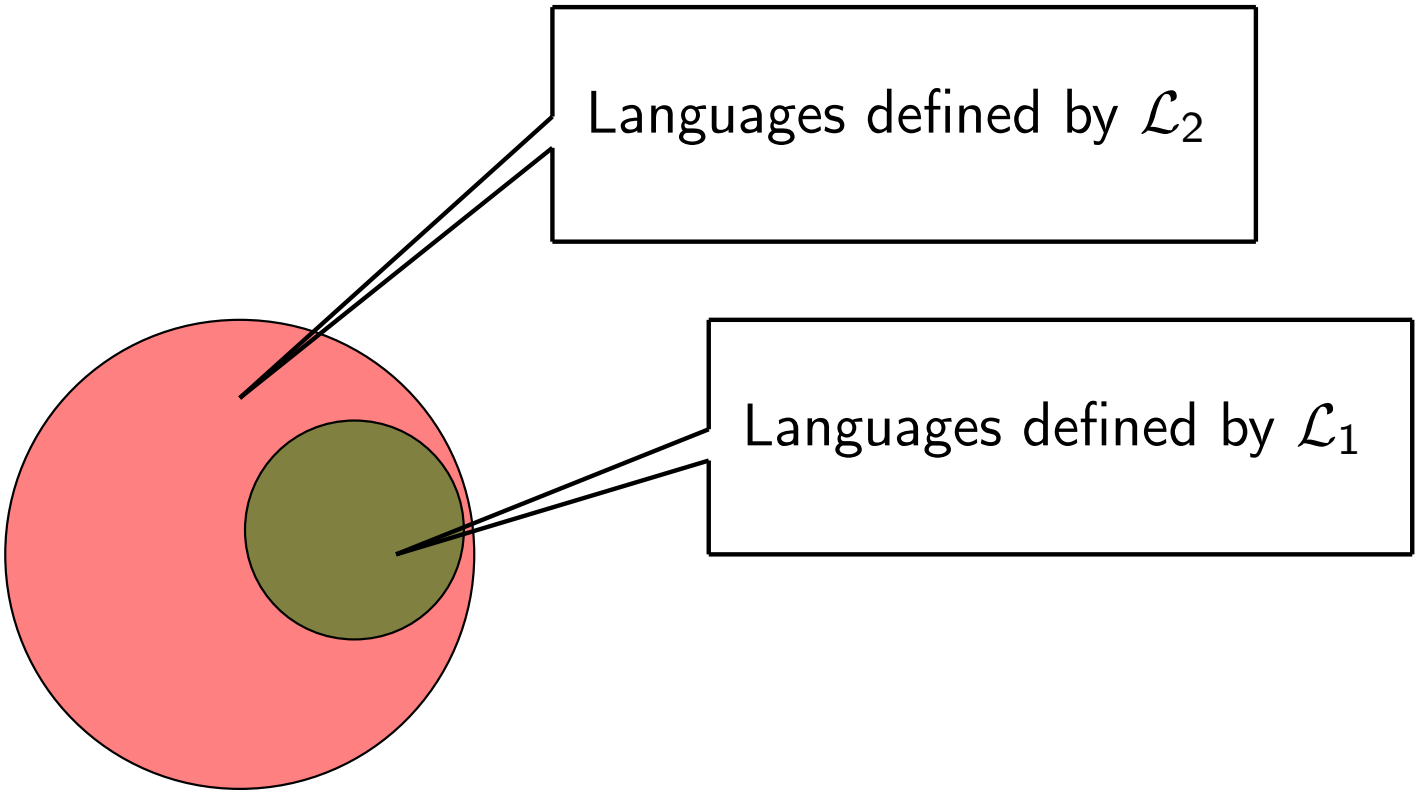
- Q: Can we express this language with 2-CNF?
- A: No.
- Proof:
  - The language accepted by φ has 7 words: all assignments other than $x_1 = x_2 = x_3 = 0$ (*false*).
  - A 2-CNF clause removes 2 assignments, which leaves us with 6 accepted words.
    E.g., $(x_1 \vee x_2)$ removes the assignments $x_1 = x_2 = x_3 = 0$ and $x_1 = x_2 = 0$, $x_3 = 1$.
  - Additional clauses only remove more assignments.

Examples



$\mathcal{L}_2$ is more expressive than $\mathcal{L}_1$. Notation: $\mathcal{L}_1 \prec \mathcal{L}_2$.

- Claim: 2-CNF $\prec$ propositional logic.
- Generally there is only a partial order between theories.

The Tradeoff

- So we see that theories can have different expressive power.
- Q: Why would we want to restrict ourselves to a theory or a fragment? Why not take some 'maximal theory'?

The Tradeoff

- So we see that theories can have different expressive power.
- Q: Why would we want to restrict ourselves to a theory or a fragment? Why not take some 'maximal theory'?
- A: Adding axioms to the theory may make it harder to decide or even undecidable.

Example: Hilbert axiom system ($\mathcal{H}$)

- Let $\mathcal{H}$ be (M.P) + the following axiom schemas:

$$\frac{}{A \rightarrow (B \rightarrow A)} \ (H1)$$

$$\frac{}{((A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))} \ (H2)$$

$$\frac{}{(\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)} \ (H3)$$

- $\mathcal{H}$ is sound and complete for propositional logic.
- This means that with $\mathcal{H}$ we can prove any valid propositional formula, and only such formulae. The proof is finite.

Example

- But there exist first order theories defined by axioms which are not sufficient for proving all T-valid formulae.

Example: First Order Peano Arithmetic

- $\Sigma = \{0, 1, +, *, =\}$
- Domain: Natural numbers
- Axioms ("semantics"):
  1. $\forall x.\ (x \neq x + 1)$
  2. $\forall x.\ \forall y.\ (x \neq y) \rightarrow (x + 1 \neq y + 1)$
  3. Induction
  4. $\forall x.\ x + 0 = x$
  5. $\forall x.\ \forall y : (x + y) + 1 = x + (y + 1)$
  6. $\forall x.\ x * 0 = 0$
  7. $\forall x.\ \forall y.\ x * (y + 1) = x * y + x$

UNDECIDABLE!

Reduction: Peano Arithmetic to Presburger Arithmetic

- $\Sigma = \{0, 1, +, \cancel{*}, =\}$
- Domain: Natural numbers
- Axioms ("semantics"):
  1. $\forall x.\ (\neq x + 1)$
  2. $\forall x.\ \forall y.\ (x \neq y) \rightarrow (x + 1 \neq y + 1)$
  3. Induction
  4. $\forall x.\ x + 0 = x$
  5. $\forall x.\ \forall y.\ (x + y) + 1 = x + (y + 1)$
  6. ~~$\forall x.\ x * 0 = 0$~~
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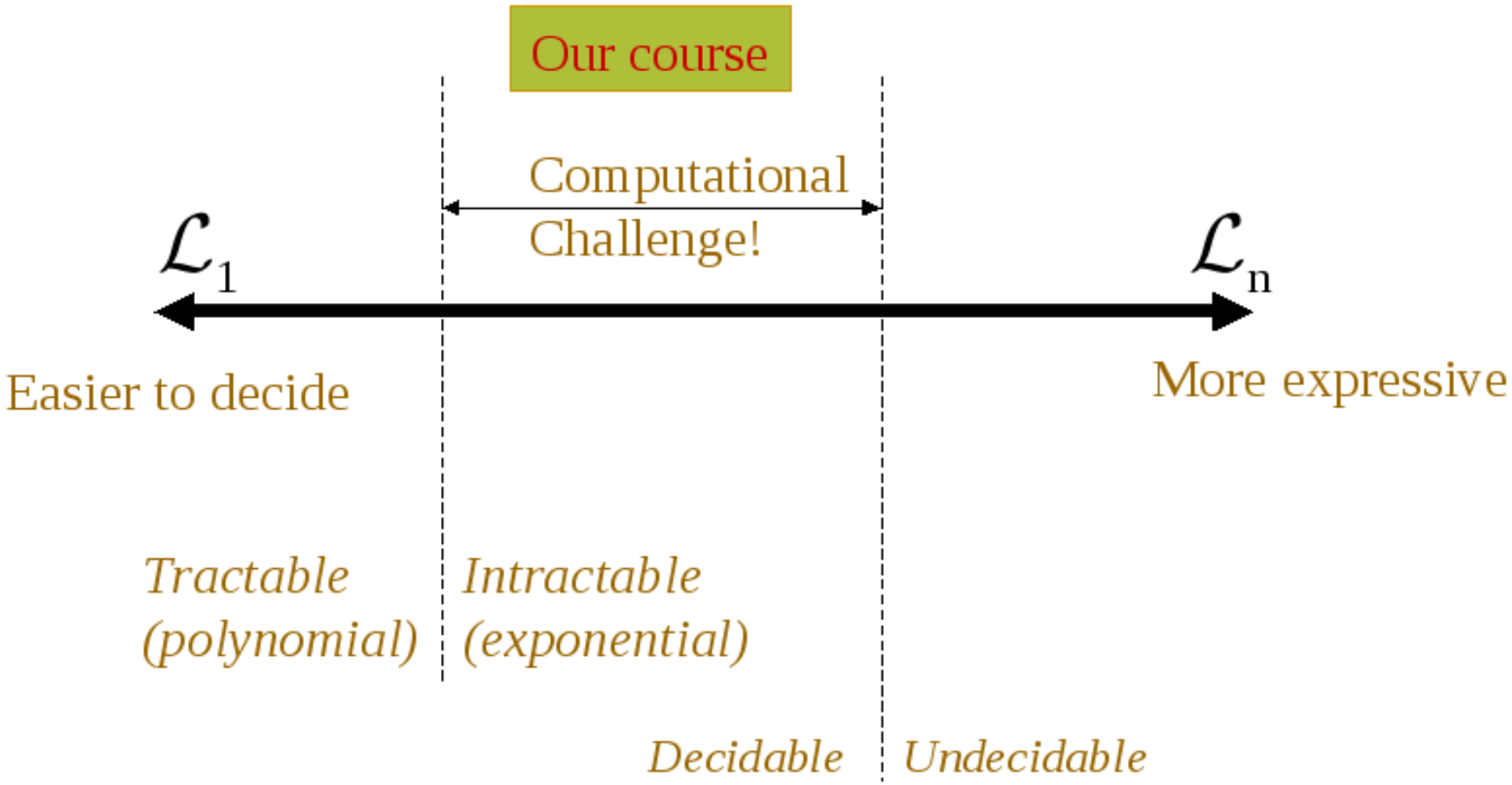
DECIDABLE!

Tradeoff: expressiveness/computational hardness



When is a specific theory useful?

- Expressible enough to state something interesting.
- Decidable (or semi-decidable) and more efficiently solvable than richer theories.
- More expressible, or more natural for expressing some models in comparison to 'leaner' theories.

Expressiveness and complexity

- Q1: Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two theories whose satisfiability problem is decidable and in the same complexity class. Is the satisfiability problem of an $\mathcal{L}_1$ formula reducible to a satisfiability problem of an $\mathcal{L}_2$ formula?

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- Q2: Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two theories whose satisfiability problems are reducible to each other. Are $\mathcal{L}_1$ and $\mathcal{L}_2$ in the same complexity class?