Satisfiability Checking Propositional Logic

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Theory of Hybrid Systems
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Propositional logic

The slides are partly taken from:

www.decision-procedures.org/slides/

Propositional logic - Outline

- Syntax of propositional logic
- Semantics of propositional logic
- Satisfiability and validity
- Modeling with propositional logic
- Normal forms
- Deductive proofs and resolution

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Syntax of propositional logic

Before we deal with satisfiability of propositional logic formulae, we must answer two questions:

- What is a propositional logic formula?
  $\rightarrow$ Syntax of propositional logic
- What is the meaning of propositional logic formulae?
  $\rightarrow$ Semantics of propositional logic

Syntax of propositional logic

- An atomic proposition is a sentence that can be either true or false.
- Propositions:
  - x is greater than y
  - Noam wrote this letter

Syntax of propositional logic

- The symbols of the language:
  - Constants: $\bot$ (false), $\top$ (true)
  - Propositional symbols (`Prop`): $a, b, c, \ldots$
  - Operators:

    Unary:

$\neg$	not

    Binary:

$\wedge$	and
$\vee$	or
$\rightarrow$	implies
$\leftrightarrow$	equivalent to
$\oplus$	xor (different than)

  - Parentheses: (,)
- Question: What is the minimal number of such symbols?
- Answer: 1 (NAND)
- For convenience, we take 2.

Formulae

- Abstract grammar of well-formed propositional formulae:

$$\varphi \;:=\; a \;\mid\; (\neg\varphi) \;\mid\; (\varphi \wedge \varphi)$$

  with $a \in \texttt{Prop}$.

- Syntactic sugar:

$$\begin{aligned}
\bot &:= (a \wedge \neg a)\\
\top &:= (a \vee \neg a)\\
(\ \varphi_1 \ \vee \ \varphi_2\ ) &:= \neg((\neg\varphi_1) \wedge (\neg\varphi_2))\\
(\ \varphi_1 \ \rightarrow \ \varphi_2\ ) &:= ((\neg\varphi_1) \vee \varphi_2)\\
(\ \varphi_1 \ \leftrightarrow \ \varphi_2\ ) &:= ((\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1))\\
(\ \varphi_1 \ \oplus \ \varphi_2\ ) &:= (\varphi_1 \leftrightarrow (\neg\varphi_2))
\end{aligned}$$

Formulae

- Examples of well-formed formulae:
  - $(\neg a)$
  - $(\neg(\neg a))$
  - $(a \wedge (b \wedge c))$
  - $(a \rightarrow (b \rightarrow c))$
- Correct expressions of propositional logic are full of unnecessary parenthesis.

Formulae

- Abbreviations:

  We write $a \text{ op } b \text{ op } c \text{ op} \ldots$

  in place of $(a \text{ op } (b \text{ op}(c \text{ op} \ldots)))$

  Thus, we write $a \wedge b \wedge c, \quad a \rightarrow b \rightarrow c, \ldots$

  in place of $(a \wedge (b \wedge c)), \quad (a \rightarrow (b \rightarrow c)), \ldots$

Formulae

- We omit parenthesis whenever we may restore them through operator precedence

binds stronger

$\longleftarrow$

$\neg \quad \wedge \quad \vee \quad \rightarrow \quad \leftrightarrow$

- Thus, we write:

$\neg\neg a$	for	$(\neg(\neg a))$,
$\neg a \wedge b$	for	$((\neg a) \wedge b)$
$a \wedge b \rightarrow c$	for	$((a \wedge b) \rightarrow c)$
$\ldots$		

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Semantics of propositional logic

- Truth tables define the semantics (=meaning) of the operators
- Convention: 0= false, 1= true

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$	$p \bigoplus q$
0	0	1	0	0	1	1	0
0	1	1	0	1	1	0	1
1	0	0	0	1	0	0	1
1	1	0	1	1	1	1	0

Question

- Question: How many binary operators can we define that have different semantics?
- Answer: 16

Assignments

- A truth-value assignment α is a mapping from variables to truth values:
  $\alpha : \texttt{Prop} \to \{0, 1\}$
- Let *Ass* denote the set of all assignments.
  Example: $\texttt{Prop} = \{a, b\}, \alpha(a) = 0, \alpha(b) = 1$
- Equivalently, we can see α as an element of $2^{\texttt{Prop}}$
  (i.e., $\alpha \in 2^{\texttt{Prop}}$ with $2^{\texttt{Prop}}$ the set of subsets of $\texttt{Prop}$).
  Meaning: α is the set of those variables that are assigned to true.
  Example: $\texttt{Prop} = \{a, b\}, \alpha = \{b\}$
- An assignment can also be seen as being of type $\alpha \in \{0, 1\}^{\texttt{Prop}}$, if we have an order on the propositions.
  Example: $\texttt{Prop} = \{a, b\}, \alpha = \{01\}$

Satisfaction relation ($\models$): Intuition

- An assignment can either satisfy or not satisfy a given formula.
- $\alpha \models \varphi$ means
  - α satisfies φ or
  - φ holds for α or
  - α is a model of φ
- We will first see an example.
- Then we will define these notions formally.

Example

- Let φ be defined as $(a \vee (b \rightarrow c))$.
- Let $\alpha : \{a, b, c\} \rightarrow \{0, 1\}$ be an assignment with $\alpha(a) = 0$, $\alpha(b) = 0$, and $\alpha(c) = 1$.

- Question: Does α satisfy φ?
  In symbols: Does it hold that $\alpha \models \varphi$?

- Answer: $(0 \vee (0 \rightarrow 1)) = (0 \vee 1) = 1$
  Hence, $\alpha \models \varphi$.

- Let us now formalize an evaluation process.

The satisfaction relation $\models$: Formalization

- $\models$ is a relation: $\models \subseteq \mathit{Ass} \times \text{Formula}$
- Examples:

$$
\begin{array}{llllll}
(\alpha, a \vee b) & \in \models & \text{or} & \alpha \models a \vee b & \text{iff} & \alpha(a) = 1 \text{ or } \alpha(b) = 1 \\
(\alpha, a \wedge b) & \in \models & \text{or} & \alpha \models a \wedge b & \text{iff} & \alpha(a) = 1 \text{ and } \alpha(b) = 1
\end{array}
$$

The satisfaction relation ($\models$): Formalization

- $\models$ is defined recursively:

$\alpha \models p$	iff $\alpha(p) = \text{true}$
$\alpha \models \neg\varphi$	iff $\alpha \not\models \varphi$
$\alpha \models \varphi_1 \wedge \varphi_2$	iff $\alpha \models \varphi_1$ and $\alpha \models \varphi_2$
$\alpha \models \varphi_1 \vee \varphi_2$	iff $\alpha \models \varphi_1$ or $\alpha \models \varphi_2$
$\alpha \models \varphi_1 \rightarrow \varphi_2$	iff $\alpha \models \varphi_1$ implies $\alpha \models \varphi_2$
$\alpha \models \varphi_1 \leftrightarrow \varphi_2$	iff $\alpha \models \varphi_1$ iff $\alpha \models \varphi_2$

From definition to an evaluation algorithm

- Truth evaluation problem:
  Given $\varphi \in$ Formula and $\alpha : \text{AP} \to \{0,1\}$, does $\alpha \models \varphi$?

```
Eval(φ, α) {
    if φ ≡ a return α(a);
    if φ ≡ (¬φ1) return ¬Eval(φ1, α);
    if φ ≡ (φ1 op φ2)
        return Eval(φ1, α) op Eval(φ2, α);
}
```

- Eval uses polynomial time and space.

It doesn't give us more than what we already know...

- Recall our example
  - Let $\varphi = (a \vee (b \rightarrow c))$
  - Let $\alpha = \{a \rightarrow 0, b \rightarrow 0, c \rightarrow 1\}$

- $$\begin{aligned} \text{Eval}(\varphi, \alpha) \; = \; & \text{Eval}(a, \alpha) \vee \text{Eval}(b \rightarrow c, \alpha) = \\ & 0 \vee (\text{Eval}(b, \alpha) \rightarrow \text{Eval}(c, \alpha)) = \\ & 0 \vee (0 \rightarrow 1) = \\ & 0 \vee 1 = \\ & 1 \end{aligned}$$

- Hence, $\alpha \models \varphi$.

We can now extend the truth table to formulae

p	q	$(p \rightarrow (q \rightarrow p))$	$(p \wedge \neg p)$
0	0	1	0
0	1	1	0
1	0	1	0
1	1	1	0

Set of assignments

- Intuition: a formula specifies a set of truth assignments.
- Remember: *Ass* denotes the set of all assignments.
- Function models : Formula $\rightarrow 2^{Ass}$

  (a formula $\rightarrow$ set of satisfying assignments)
- Recursive definition:
  - $\text{models}(a) = \{\alpha \mid \alpha(a) = 1\}$, $a \in \texttt{Prop}$
  - $\text{models}(\neg\varphi_1) = Ass \setminus \text{models}(\varphi_1)$
  - $\text{models}(\varphi_1 \wedge \varphi_2) = \text{models}(\varphi_1) \cap \text{models}(\varphi_2)$
  - $\text{models}(\varphi_1 \vee \varphi_2) = \text{models}(\varphi_1) \cup \text{models}(\varphi_2)$
  - $\text{models}(\varphi_1 \rightarrow \varphi_2) = (Ass \setminus \text{models}(\varphi_1)) \cup \text{models}(\varphi_2)$

Example

- $\text{models}(a \vee b) = \{\alpha \in \mathit{Ass} \mid \alpha(a) = 1 \text{ or } \alpha(b) = 1\}$
- This is compatible with the recursive definition:

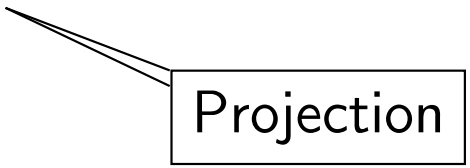
$$\text{models}(a \vee b) = \mathit{models}(a) \cup \mathit{models}(b) =$$
$$\{\alpha \in \mathit{Ass} \mid \alpha(a) = 1\} \cup \{\alpha \in \mathit{Ass} \mid \alpha(b) = 1\}$$

Theorem

- Let $\varphi \in$ Formula and $\alpha \in \mathit{Ass}$, then the following statements are equivalent:
  1. $\alpha \models \varphi$
  2. $\alpha \in \mathsf{models}(\varphi)$

Only the projected assignment matters...

- AP(φ) - the atomic propositions in φ.
- Clearly AP(φ) $\subseteq$ `Prop`.
- Let $\alpha_1, \alpha_2 \in$ *Ass* and $\varphi \in$ Formula.
- Lemma: if $\alpha_1|_{\mathsf{AP}(\varphi)} = \alpha_2|_{\mathsf{AP}(\varphi)}$, then



$\alpha_1 \models \varphi$ iff $\alpha_2 \models \varphi$

Corollary: $\alpha \models \varphi$ iff $\alpha|_{\mathsf{AP}(\varphi)} \models \varphi$

- We will assume, for simplicity, that `Prop` $=$ AP(φ).

Extension of $\models$ to sets of assignments

- Let $\varphi \in$ Formula.
- Let T be a set of assignments, i.e., $T \subseteq 2^{Ass}$
- Definition: $\models \subseteq 2^{Ass} \times$ Formula with

$$T \models \varphi \text{ iff } T \subseteq \text{models}(\varphi)$$

Extension of $\models$ to formulae

- $\models \subseteq 2^{\text{Formula}} \times 2^{\text{Formula}}$

- Definition. Let φ_1, φ_2 be propositional formulae.

  $\varphi_1 \models \varphi_2$

  iff models(φ_1) $\subseteq$ models(φ_2), or equivalently
  iff for all $\alpha \in Ass$

  if $\alpha \models \varphi_1$ then $\alpha \models \varphi_2$

  Examples:

  $x_1 \wedge x_2 \models x_1 \vee x_2$

  $x_1 \wedge x_2 \models x_2 \vee x_3$

Short summary for propositional logic

- Syntax: $\varphi \ := \ \texttt{prop} \mid (\neg\varphi) \mid (\varphi \wedge \varphi)$
- Semantics:
  - Assignments:

$$\alpha : \texttt{Prop} \to \{0,1\}$$
$$\alpha \in 2^{\texttt{Prop}}$$
$$\alpha \in \{0,1\}^{\texttt{Prop}}$$

  - Satisfiability relation:

$$\begin{array}{lll} \models \subseteq \mathit{Ass} \times \text{Formula} & , & (\text{e.g., } \alpha \models \varphi) \\ \models \subseteq 2^{\mathit{Ass}} \times \text{Formula} & , & (\text{e.g., } T \models \varphi) \\ \models \subseteq \text{Formula} \times \text{Formula} & , & (\text{e.g., } \varphi_1 \models \varphi_2) \\ \text{models} : \ \text{Formula} \to 2^{\mathit{Ass}} & , & (\text{e.g., } \ \text{models}(\varphi)) \end{array}$$

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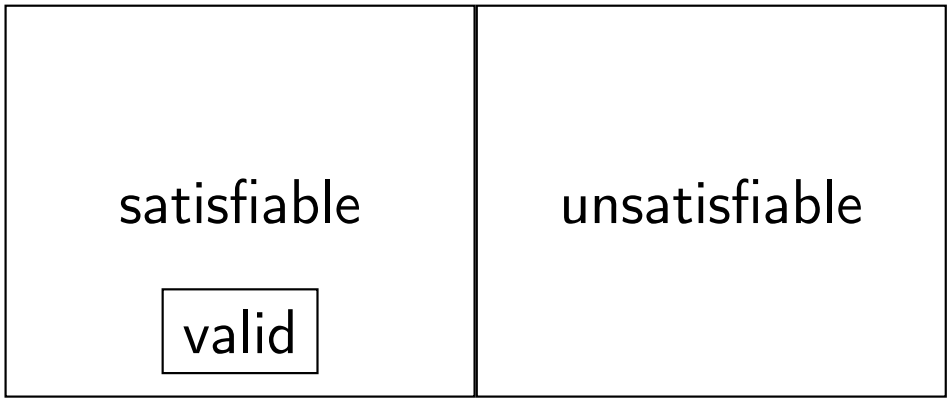
Semantic classification of formulae

- A formula φ is called valid if models(φ) = Ass.
  (Also called a tautology).
- A formula φ is called satisfiable if models(φ) $\neq \emptyset$.
- A formula φ is called unsatisfiable if models(φ) = $\emptyset$.
  (Also called a contradiction).



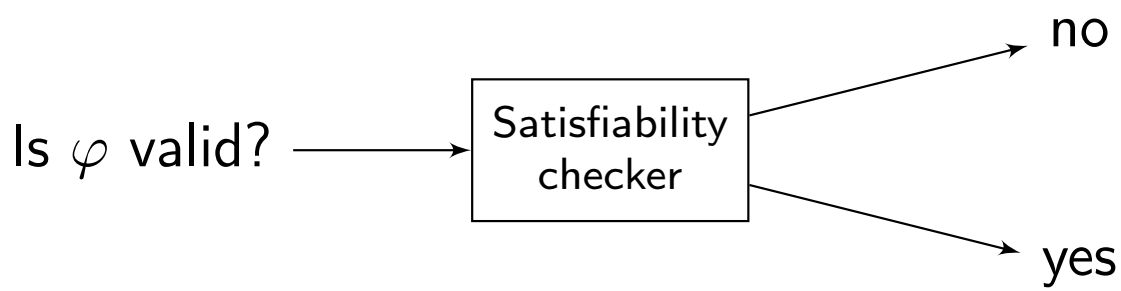
Validity and satisfiability

p	q	$(p \rightarrow (q \rightarrow q))$	$(p \wedge \neg p)$	$p \vee \neg q$
0	0	1	0	1
0	1	1	0	0
1	0	1	0	1
1	1	1	0	1

Characteristics of formulae

Lemma:

- A formula φ is valid iff $\neg\varphi$ is unsatisfiable
- φ is satisfiable iff $\neg\varphi$ is not valid



Look what we can do now...

- We can write:
  - $\models \varphi$ when φ is valid
  - $\not\models \varphi$ when φ is not valid
  - $\not\models \neg\varphi$ when φ is satisfiable
  - $\models \neg\varphi$ when φ is unsatisfiable

Examples

- $(x_1 \wedge x_2) \rightarrow (x_1 \vee x_2)$ is valid
- $(x_1 \vee x_2) \rightarrow x_1$ is satisfiable
- $(x_1 \wedge x_2) \wedge \neg x_1$ is unsatisfiable

Time for equivalences

- Here are some valid formulae:
  - $\models a \wedge 1 \leftrightarrow a$
  - $\models a \wedge 0 \leftrightarrow 0$
  - $\models \neg\neg a \leftrightarrow a$ // The double-negation rule
  - $\models a \wedge (b \vee c) \leftrightarrow (a \wedge b) \vee (a \wedge c)$
- Some more (De Morgan rules):
  - $\models \neg(a \wedge b) \leftrightarrow (\neg a \vee \neg b)$
  - $\models \neg(a \vee b) \leftrightarrow (\neg a \wedge \neg b)$

The decision problem of formulae

- The decision problem:

  Given a propositional formula φ, is φ satisfiable?

- An algorithm that always terminates with a correct answer to this problem is called a decision procedure for propositional logic.

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Before we solve this problem...

- Suppose we can solve the satisfiability problem... how can this help us?
- There are numerous problems in the industry that are solved via the satisfiability problem of propositional logic
  - Logistics
  - Planning
  - Electronic Design Automation industry
  - Cryptography
  - ...

Example 1: Placement of wedding guests

- Three chairs in a row: $1, 2, 3$
- We need to place Aunt, Sister and Father.
- Constraints:
  - Aunt doesn't want to sit near Father
  - Aunt doesn't want to sit in the left chair
  - Sister doesn't want to sit to the right of Father

- Question: Can we satisfy these constraints?

Example 1 (continued)

- Denote: Aunt = 1, Sister = 2, Father = 3
- Introduce a propositional variable for each pair (person, place).
- x_{ij} = "person i is sited in place j, for $1 \leq i, j \leq 3$"
- Constraints:
  - Aunt doesn't want to sit near Father:
    $((x_{1,1} \vee x_{1,3}) \rightarrow \neg x_{3,2}) \wedge (x_{1,2} \rightarrow (\neg x_{3,1} \wedge \neg x_{3,3}))$
  - Aunt doesn't want to sit in the left chair
    $\neg x_{1,1}$
  - Sister doesn't want to sit to the right of Father
    $x_{3,1} \rightarrow \neg x_{2,2} \wedge x_{3,2} \rightarrow \neg x_{2,3}$

Example 1 (continued)

- More constraints:
- Each person is placed:

  $(x_{1,1} \vee x_{1,2} \vee x_{1,3}) \wedge (x_{2,1} \vee x_{2,2} \vee x_{2,3}) \wedge (x_{3,1} \vee x_{3,2} \vee x_{3,3})$
- Or, more concisely:

$$\bigwedge_{i=1}^{3} \bigvee_{j=1}^{3} x_{i,j}$$

- No person is placed in more than one place:

$$\bigwedge_{i=1}^{3} \bigwedge_{j=1}^{2} \bigwedge_{k=j+1}^{3} (\neg x_{i,j} \vee \neg x_{i,k})$$

- Overall 9 variables, 26 conjoined constraints.

Example 2: Assignment of frequencies

- n radio stations
- For each assign one of k transmission frequencies, $k < n$.
- E – set of pairs of stations, that are too close to have the same frequency.

- Question: Can we assign to each station a frequency, such that no statin pairs from E have the same frequency?

Example 2 (continued)

- $x_{i,j}$: station i is assigned frequency j, for $1 \leq i \leq n$, $1 \leq j \leq k$.
  - Every station is assigned at least one frequency:

$$\bigwedge_{i=1}^{n} \bigvee_{j=1}^{k} x_{i,j}$$

  - Every station is assigned not more than one frequency:

$$\bigwedge_{i=1}^{n} \bigwedge_{j=1}^{k-1} (x_{i,j} \rightarrow \bigwedge_{j<t\leq k} \neg x_{i,t})$$

  - Close stations are not assigned the same frequency:
    For each $(i,j) \in E$,

$$\bigwedge_{t=1}^{k} (x_{i,t} \rightarrow \neg x_{j,t})$$

Two classes of algorithms for validity

- Question: Is φ satisfiable? (Is $\neg\varphi$ valid?)
- Complexity: NP-Complete (Cook's theorem)
- Two classes of algorithms for finding out:
  - Enumeration of possible solutions (Truth tables etc.)
  - Deduction

- More generally (beyond propositional logic):
  - Enumeration is possible only in some logics.
  - Deduction cannot necessarily be fully automated.

The satisfiability problem: Enumeration the first

- Given a formula φ, is φ satisfiable?

```
Boolean SAT(φ){
        result:=false;
        for all α ∈ Ass
          result = result ∨ Eval(φ, α);
        return result;
}
```

The satisfiability problem: Enumeration the second

- Given a formula φ, is φ satisfiable?
- Use substitution to eliminate all variables one by one:

$$\varphi \quad \text{iff} \quad \varphi[0/a] \vee \varphi[1/a]$$

- There must be a better way to do that in practice.

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Definitions

- Definition: A literal is either an atom or a negation of an atom.
- Let $\varphi = \neg(a \vee \neg b)$. Then:
- Atoms: $\mathrm{AP}(\varphi) = \{a, b\}$
- Literals: $\mathrm{lit}(\varphi) = \{a, \neg b\}$
- Equivalent formulae can have different literals
- $\varphi' = \neg a \wedge b$
- Now $\mathrm{lit}(\varphi') = \{\neg a, b\}$

Definitions

- Definition: a term is a conjunction of literals
  - Example: $(a \wedge \neg b \wedge c)$
- Definition: a clause is a disjunction of literals
  - Example: $(a \vee \neg b \vee c)$

Negation Normal Form (NNF)

- Definition: A formula is in Negation Normal Form (NNF) iff
  (1) it contains only $\neg$, $\wedge$ and $\vee$ as connectives and
  (2) only atoms are negated.

- Examples:
- $\varphi_1 = \neg(a \vee \neg b)$ is not in NNF
- $\varphi_2 = \neg a \wedge b$ is in NNF

Converting to NNF

- Every formula can be converted to NNF in linear time:
  - Eliminate all connectives other than $\wedge$, $\vee$, $\neg$
  - Use De Morgan and double-negation rules to push negations to the right

- Example: $\varphi = \neg(a \rightarrow \neg b)$
  - Eliminate '$\rightarrow$' : $\varphi = \neg(\neg a \vee \neg b)$
  - Push negation using De Morgan: $\varphi = (\neg\neg a \wedge \neg\neg b)$
  - Use double-negation rule: $\varphi = (a \wedge b)$

Disjunctive Normal Form (DNF)

- Definition: A formula is said to be in Disjunctive Normal Form (DNF) iff it is a disjunction of terms.
  - In other words, it is a formula of the form

$$\bigvee_i (\bigwedge_j l_{i,j})$$

where $l_{i,j}$ is the j-th literal in the i-th term.

- Example:

$$\varphi = (a \wedge \neg b \wedge c) \vee (\neg a \wedge d) \vee (b) \quad \text{is in DNF}$$

- DNF is a special case of NNF

Converting to DNF

- Every formula can be converted to DNF in exponential time and space:
  1. Convert to NNF
  2. Distribute disjunctions following the rule:
     $\models a \wedge (b \vee c) \leftrightarrow ((a \wedge b) \vee (a \wedge c))$
- Example:

$$\begin{aligned}\varphi &= (a \vee b) \wedge (\neg c \vee d) \\ &= ((a \vee b) \wedge (\neg c)) \vee ((a \vee b) \wedge d) \\ &= (a \wedge \neg c) \vee (b \wedge \neg c) \vee (a \wedge d) \vee (b \wedge d)\end{aligned}$$

- Question: How many clauses would the DNF have had if we started from a conjunction of n binary clauses (i.e., clauses with 2 literals)?

Satisfiability of DNF

- Is the following DNF formula satisfiable?

  $(a_1 \wedge a_2 \wedge \neg a_1) \vee (a_2 \wedge a_1) \vee (a_2 \wedge \neg a_3 \wedge a_3)$

- Question: What is the complexity of the satisfiability check of DNF formulae?

Conjunctive Normal Form (CNF)

- Definition: A formula is said to be in Conjunctive Normal Form (CNF) iff it is a conjunction of clauses.

  In other words, it is a formula of the form

$$\bigwedge_i \left(\bigvee_j l_{i,j}\right)$$

  where $l_{i,j}$ is the j-th literal in the i-th clause.

- Example:

$$\varphi = (a \vee \neg b \vee c) \wedge (\neg a \vee d) \wedge (b) \quad \text{is in CNF}$$

- CNF is a special case of NNF

Converting to CNF

- Every formula can be converted to CNF:
  - in exponential time and space with the same set of atoms, or
  - in linear time and space if new variables are added.
- For the latter—the so-called Tseitin's encoding—the original and the converted formulae are equi-satisfiable, but not equivalent.

- Question: Can there be any such linear transformation into DNF?
- Answer: No. Linear DNF transformation and linear DNF solution would violate the NP-completeness of the problem.

Converting to CNF: The exponential way

CNF(φ){

case

φ is a literal: return φ

φ is $\varphi_1 \wedge \varphi_2$: return CNF($\varphi_1$) $\wedge$ CNF(φ_2)

φ is $\varphi_1 \vee \varphi_2$: return Dist(CNF(φ_1),CNF(φ_2))

}

Dist(φ_1,φ_2) {

case

φ_1 is $\varphi_{11} \wedge \varphi_{12}$: return Dist($\varphi_{11}$,$\varphi_2$) $\wedge$ Dist(φ_{12},φ_2)

φ_2 is $\varphi_{21} \wedge \varphi_{22}$: return Dist($\varphi_1$,$\varphi_{21}$) $\wedge$ Dist(φ_1,φ_{22})

else: return $\varphi_1 \vee \varphi_2$

}

Converting to CNF: The exponential way

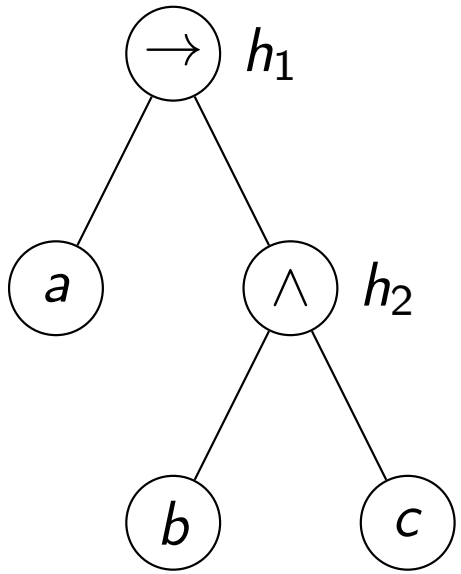
- Consider the formula
- $\varphi = (a_1 \wedge b_1) \vee (a_2 \wedge b_2)$
- $\text{CNF}(\varphi) = (a_1 \vee a_2) \wedge (a_1 \vee b_2) \wedge (b_1 \vee a_2) \wedge (b_1 \vee b_2)$

- Now consider: $\varphi_n = (a_1 \wedge b_1) \vee (a_2 \wedge b_2) \vee \ldots \vee (a_n \wedge b_n)$
- Question: How many clauses does $\text{CNF}(\varphi)$ return?
- Answer: 2^n

Converting to CNF: Tseitin's encoding

- Consider the formula
  $\varphi = (a \rightarrow (b \wedge c))$

The Parse Tree:



- Associate a new auxiliary variable with each gate.
- Add constraints that define these new variables.
- Finally, enforce the root node.

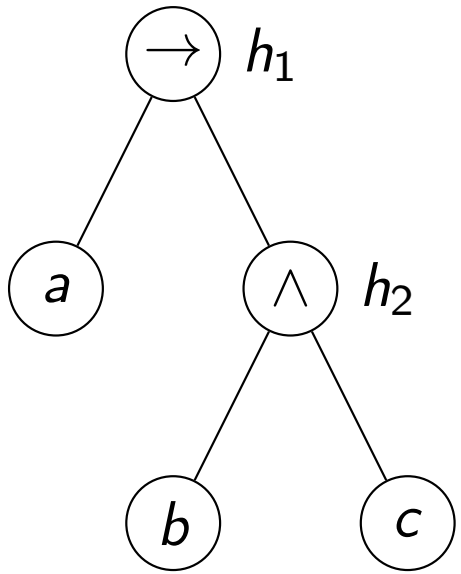
Converting to CNF: Tseitin's encoding

- Need to satisfy:

  $(h_1 \leftrightarrow (a \rightarrow h_2)) \wedge$

  $(h_2 \leftrightarrow (b \wedge c)) \wedge$

  (h_1)



  - Each gate encoding has a CNF representation with 3 or 4 clauses.

Converting to CNF: Tseitin's encoding

- Need to satisfy:
  $(h_1 \leftrightarrow (a \rightarrow h_2)) \wedge (h_2 \leftrightarrow (b \wedge c)) \wedge (h_1)$

- First: $(h_1 \vee a) \wedge (h_1 \vee \neg h_2) \wedge (\neg h_1 \vee \neg a \vee h_2)$
- Second: $(\neg h_2 \vee b) \wedge (\neg h_2 \vee c) \wedge (h_2 \vee \neg b \vee \neg c)$

Converting to CNF: Tseitin's encoding

- Let's go back to

  $\varphi_n = (x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee \cdots \vee (x_n \wedge y_n)$

- With Tseitin's encoding we need:
  - n auxiliary variables $a_1, \ldots, a_n$.
  - Each adds 3 constraints.
  - Top clause: $(a_1 \vee \cdots \vee a_n)$

- Hence, we have
  - $3n + 1$ clauses, instead of 2^n.
  - $3n$ variables rather than $2n$.

What now?

- Time to solve the decision problem for propositional logic.
  - The only algorithm we saw so far was building truth tables.

Two classes of algorithms for validity

- Question: Is φ valid?
  - Equivalently: is $\neg\varphi$ satisfiable?
- Two classes of algorithm for finding out:
  1. Enumeration of possible solutions (Truth tables etc.)
  2. Deduction
- In general (beyond propositional logic):
  - Enumeration is possible only in some theories.
  - Deduction typically cannot be fully automated.

The satisfiability problem: Enumeration

- Given a formula φ, is φ satisfiable?

```
Boolean SAT(φ) {
    result := false;
    for all α ∈ Ass
        result = result ∨ Eval(φ, α);
    return result;
}
```

- NP-Complete (Cook's theorem)

Propositional logic - Outline

- Syntax of propositional logic
- Semantics of propositional logic
- Satisfiability and validity
- Modeling with propositional logic
- Normal forms
- Deductive proofs and resolution

Deduction requires axioms and inference rules

- Inference rules:

$$\frac{\text{Antecedents}}{\text{Consequents}} \quad \text{(rule-name)}$$

Meaning: If all antecedents hold then at least one of the consequents can be derived.

- Examples:

$$\frac{a \to b \qquad b \to c}{a \to c} \quad \text{(Trans)}$$

$$\frac{a \to b \qquad a}{b} \quad \text{(M.P.)}$$

Axioms

- Axioms are inference rules with no antecedents, e.g.,

$$\frac{}{a \to (b \to a)} \quad (\text{H1})$$

- We can turn an inference rule into an axiom if we have '$\to$' in the logic.
- So the difference between them is not sharp.

Proofs

- A proof uses a given set of axioms and inference rules.
- This is called the proof system.
- Let $\mathcal{H}$ be a proof system.
- $\Gamma \vdash_{\mathcal{H}} \varphi$ means: There is a proof of φ in system $\mathcal{H}$ whose premises are included in Γ
- $\vdash_{\mathcal{H}}$ is called the provability relation.

Example

- Let $\mathcal{H}$ be the proof system comprised of the rules Trans and M.P. that we saw earlier:

$$\frac{a \to b \quad b \to c}{a \to c} \qquad \text{(Trans)}$$

$$\frac{a \to b \quad a}{b} \qquad \text{(M.P.)}$$

- Does the following relation hold?

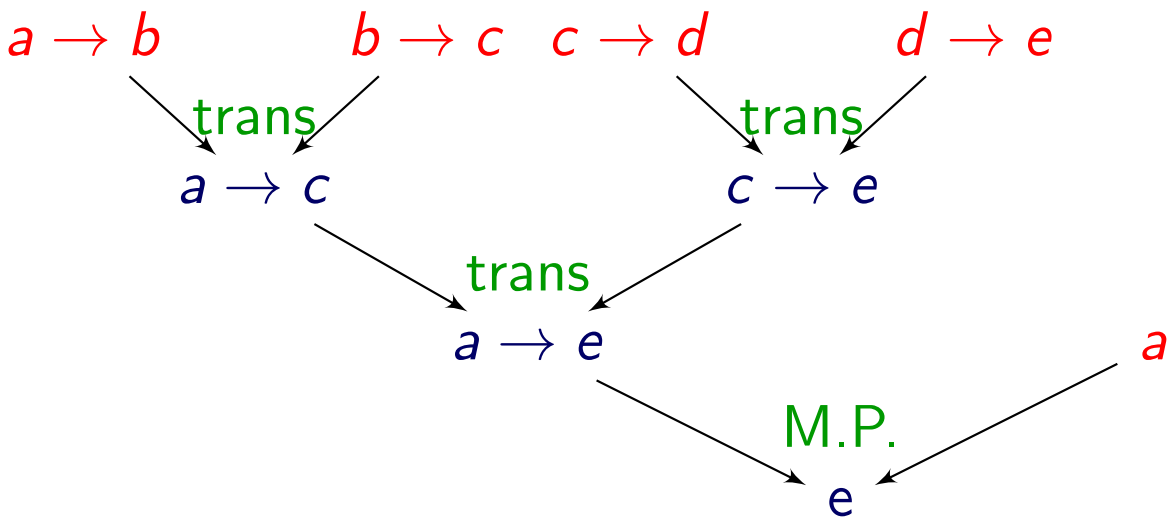
$$a \to b,\ b \to c,\ c \to d,\ d \to e,\ a \quad \vdash_{\mathcal{H}} \quad e$$

Deductive proof: Example

$$a \to b,\ b \to c,\ c \to d,\ d \to e,\ a \quad \vdash_{\mathcal{H}} \quad e$$

1.	$a \to b$	premise
2.	$b \to c$	premise
3.	$a \to c$	1, 2, Trans
4.	$c \to d$	premise
5.	$d \to e$	premise
6.	$c \to e$	4, 5, Trans
7.	$a \to e$	3, 6, Trans
8.	a	premise
9.	e	7, 8, M.P.

Proof graph (DAG)



Correctness and Completeness

- $\vdash$ is a relation defined by syntactic transformations of the underlying proof system.
- For a given proof system $\mathcal{H}$,
  - Correctness: Does $\vdash$ conclude "correct" conclusions from premises?
  - Completeness: Can we conclude all true statements with $\mathcal{H}$?
- Correct with respect to what?
- With respect to the semantic definition of the logic. In the case of propositional logic truth tables give us this.

Soundness and completeness

- Let $\mathcal{H}$ be a proof system

$$\begin{array}{lllll} \text{Soundness of } \mathcal{H}: & \text{if} & \vdash_{\mathcal{H}} \varphi & \text{then} & \models \varphi \\ \text{Completeness of } \mathcal{H}: & \text{if} & \models \varphi & \text{then} & \vdash_{\mathcal{H}} \varphi \end{array}$$

- How to prove soundness and completeness?

Example: Hilbert axiom system (H)

- Let H be (M.P.) together with the following axiom schemes:

$$\frac{}{a \to (b \to a)} \quad (H1)$$

$$\frac{}{((a \to (b \to c)) \to ((a \to b) \to (a \to c)))} \quad (H2)$$

$$\frac{}{(\neg b \to \neg a) \to (a \to b)} \quad (H3)$$

- H is sound and complete

Soundness and completeness

- To prove soundness of H, prove the soundness of its axioms and inference rules (easy with truth-tables).
  For example:

a	b	$a \to (b \to a)$
0	0	1
0	1	1
1	0	1
1	1	1

- Completeness: harder, but possible.

The resolution inference system

- The resolution inference rule for CNF:

$$\frac{(l \vee l_1 \vee l_2 \vee \ldots \vee l_n) \quad (\neg l \vee l'_1 \vee \ldots \vee l'_m)}{(l_1 \vee \ldots \vee l_n \vee l'_1 \vee \ldots \vee l'_m)} \text{ Resolution}$$

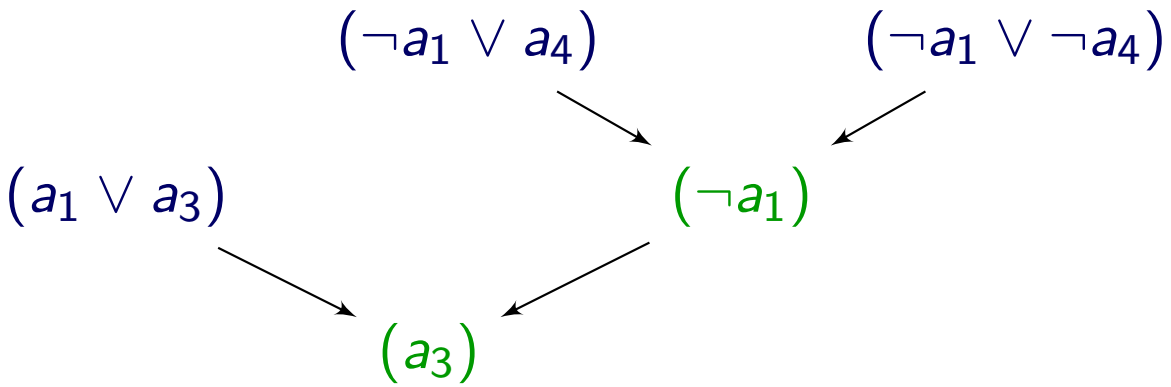
- Example:

$$\frac{(a \vee b) \quad (\neg a \vee c)}{(b \vee c)}$$

- We first see some example proofs, before proving soundness and completeness.

Proof by resolution

- Let $\varphi = (a_1 \vee a_3) \wedge (\neg a_1 \vee a_2 \vee a_5) \wedge (\neg a_1 \vee a_4) \wedge (\neg a_1 \vee \neg a_4)$
- We'll try to prove $\varphi \rightarrow (a_3)$

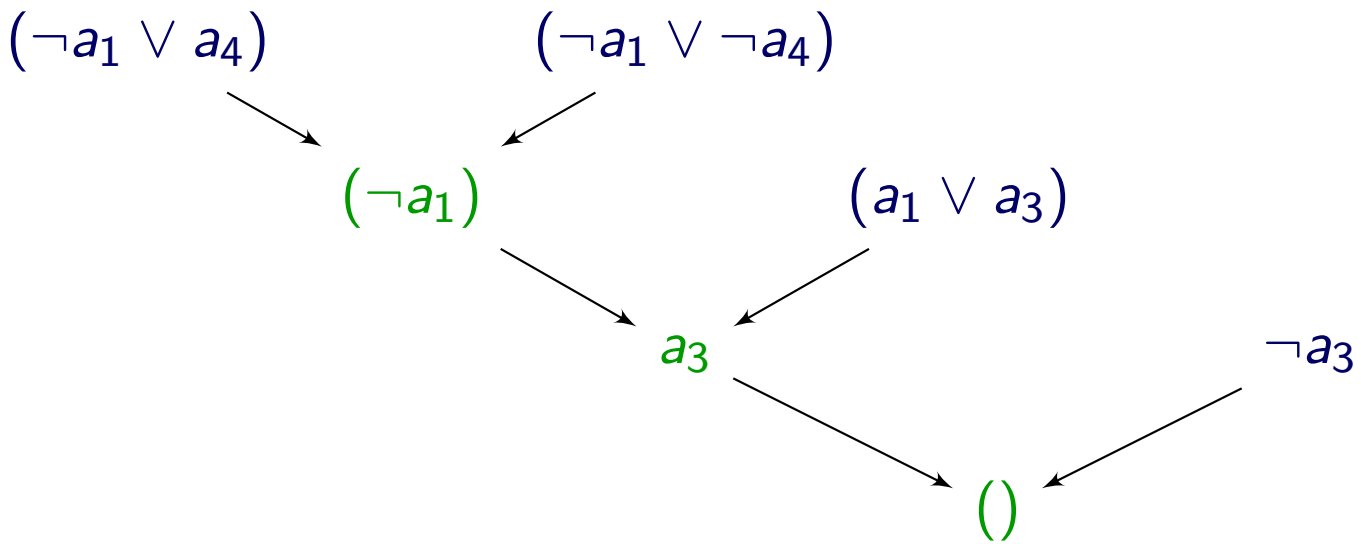


Resolution

- Resolution is a sound and complete inference system for CNF.
- If the input formula is unsatisfiable, there exists a proof of the empty clause.

Example

Let $\varphi = (a_1 \vee a_3) \wedge (\neg a_1 \vee a_2) \wedge (\neg a_1 \vee a_4) \wedge (\neg a_1 \vee \neg a_4) \wedge (\neg a_3)$.



Soundness and completeness of resolution

- Soundness is straightforward. Just prove by truth table that

$$\models ((\varphi_1 \vee a) \wedge (\varphi_2 \vee \neg a)) \rightarrow (\varphi_1 \vee \varphi_2).$$

- Completeness is a bit more involved.
  Basic idea: Use resolution for variable elimination .

$$\begin{gathered}(a \vee \varphi_1) \wedge \ldots \wedge (a \vee \varphi_n) \wedge \\ (\neg a \vee \psi_1) \wedge \ldots (\neg a \vee \psi_m) \wedge \\ R \\ \Leftrightarrow \\ (\varphi_1 \vee \psi_1) \wedge \ldots \wedge (\varphi_1 \vee \psi_m) \wedge \\ \ldots \\ (\varphi_n \vee \psi_1) \wedge \ldots (\varphi_n \vee \psi_m) \wedge \\ R\end{gathered}$$

  where φ_i $(i = 1, \ldots, n)$, ψ_j $(j = 1, \ldots, m)$, and R contain neither a nor $\neg a$.