

Propositional logic on examples

Satisfiability with truth table

Example formula:

$$\phi := \neg(a \rightarrow (b \vee \neg c))$$

a	b	c	$a \rightarrow (b \vee \neg c)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

Satisfiability with semantical algorithm

$$\begin{aligned}\text{Eval}(\alpha, p) &= \alpha(p) \\ \text{Eval}(\alpha, \neg A) &= \neg \text{Eval}(\alpha, A) \\ \text{Eval}(\alpha, A \vee B) &= \text{Eval}(\alpha, A) \vee \text{Eval}(\alpha, B) \\ \text{Eval}(\alpha, A \wedge B) &= \text{Eval}(\alpha, A) \wedge \text{Eval}(\alpha, B) \\ \text{Eval}(\alpha, A \rightarrow B) &= \text{Eval}(\alpha, \neg A) \vee \text{Eval}(\alpha, B) \\ \text{Eval}(\alpha, A \leftrightarrow B) &= \text{Eval}(\alpha, A \rightarrow B) \wedge \text{Eval}(\alpha, A \leftarrow B)\end{aligned}$$

Satisfiability with semantical algorithm

Example formula:

$$\phi := \neg(a \rightarrow (b \vee \neg c))$$

- ▶ $\alpha : a = 0, b = 0, c = 0$:
- ▶ $\text{Eval}(\alpha, \neg(a \rightarrow (b \vee \neg c))) = \text{Eval}(\alpha, a \rightarrow (b \vee \neg c))$
 - $\text{Eval}(\alpha, a \rightarrow (b \vee \neg c)) = \text{Eval}(\alpha, \neg a) \vee \text{Eval}(\alpha, b \vee \neg c)$
 - $\text{Eval}(\alpha, \neg a) = \neg \text{Eval}(\alpha, a) = 1$
 - $\text{Eval}(\alpha, b \vee \neg c) = \text{Eval}(\alpha, b) \vee \text{Eval}(\alpha, \neg c)$
 - $\text{Eval}(\alpha, b) = 0$
 - $\text{Eval}(\alpha, \neg c) = \neg \text{Eval}(\alpha, c) = 1$
 - $\text{Eval}(\alpha, b \vee \neg c) = 1$
 - $\text{Eval}(\alpha, a \rightarrow (b \vee \neg c)) = 1$
 - $\text{Eval}(\alpha, \neg(a \rightarrow (b \vee \neg c))) = 0$

NNF conversion

Only operators $\neg, \vee, \wedge$, negation only in front of atomic propositions.

Example formula:

$$\phi := \neg(a \rightarrow (b \vee \neg c))$$

$$\begin{aligned} & \neg(a \rightarrow (b \vee \neg c)) \\ = & \neg(\neg a \vee (b \vee \neg c)) \\ = & \neg(\neg a \vee b \vee \neg c) \\ = & a \wedge \neg b \wedge c \end{aligned}$$

CNF conversion: The exponential way

CNF: $\bigwedge_{i=1,\dots,n} \bigvee_{j=1,\dots,m} l_{ij}$

Example formula:

$$\phi := (a \wedge b) \vee (\neg c \wedge (d \vee e))$$

$$\begin{aligned} & (a \wedge b) \vee (\neg c \wedge (d \vee e)) \\ = & (a \vee (\neg c \wedge (d \vee e))) \wedge (b \vee (\neg c \wedge (d \vee e))) \\ = & (a \vee (\neg c \wedge d) \vee (\neg c \wedge e)) \wedge (b \vee (\neg c \wedge d) \vee (\neg c \wedge e)) \\ = & (a \vee \neg c \vee (\neg c \wedge e)) \wedge (a \vee d \vee (\neg c \wedge e)) \wedge \\ & (b \vee \neg c \vee (\neg c \wedge e)) \wedge (b \vee d \vee (\neg c \wedge e)) \\ = & (a \vee \neg c \vee \neg c) \wedge (a \vee \neg c \vee e) \wedge (a \vee d \vee \neg c) \wedge (a \vee d \vee e) \wedge \\ & (b \vee \neg c \vee \neg c) \wedge (b \vee \neg c \vee e) \wedge (b \vee d \vee \neg c) \wedge (b \vee d \vee e) \end{aligned}$$

CNF conversion: Tseitin's encoding

Example formula:

$$\phi := (a \wedge b) \vee (\neg c \wedge (d \vee e))$$

- ▶ $a_1 \leftrightarrow (a_2 \vee a_3)$
- ▶ $a_2 \leftrightarrow (a \wedge b)$
- ▶ $a_3 \leftrightarrow (\neg c \wedge a_4)$
- ▶ $a_4 \leftrightarrow (d \vee e)$
- ▶ a_1

CNF conversion: Tseitin's encoding

- ▶ $h \leftrightarrow (p_1 \vee p_2)$
 $= (h \rightarrow (p_1 \vee p_2)) \quad \wedge \quad (h \leftarrow (p_1 \vee p_2))$
 $= (\neg h \vee (p_1 \vee p_2)) \quad \wedge \quad (h \vee \neg(p_1 \vee p_2))$
 $= (\neg h \vee p_1 \vee p_2) \quad \wedge \quad (h \vee (\neg p_1 \wedge \neg p_2))$
 $= (\neg h \vee p_1 \vee p_2) \quad \wedge \quad (h \vee \neg p_1) \quad \wedge \quad (h \vee \neg p_2)$
- ▶ $h \leftrightarrow (p_1 \wedge p_2)$
 $= (h \rightarrow (p_1 \wedge p_2)) \quad \wedge \quad (h \leftarrow (p_1 \wedge p_2))$
 $= (\neg h \vee (p_1 \wedge p_2)) \quad \wedge \quad (h \vee \neg(p_1 \wedge p_2))$
 $= (\neg h \vee p_1) \quad \wedge \quad (\neg h \vee p_2) \quad \wedge \quad (h \vee \neg p_1 \vee \neg p_2)$

CNF conversion: Tseitin's encoding

$$\phi := (a \wedge b) \vee (\neg c \wedge (d \vee e))$$

$h \leftrightarrow (p_1 \vee p_2) = (\neg h \vee p_1 \vee p_2) \wedge (h \vee \neg p_1) \wedge (h \vee \neg p_2)$
$h \leftrightarrow (p_1 \wedge p_2) = (\neg h \vee p_1) \wedge (\neg h \vee p_2) \wedge (h \vee \neg p_1 \vee \neg p_2)$

$a_1 \leftrightarrow (a_2 \vee a_3)$	$= (\neg a_1 \vee a_2 \vee a_3)$	$\wedge (a_1 \vee \neg a_2)$	$\wedge (a_1 \vee \neg a_3)$
$a_2 \leftrightarrow (a \wedge b)$	$= (\neg a_2 \vee a)$	$\wedge (\neg a_2 \vee b)$	$\wedge (a_2 \vee \neg a \vee \neg b)$
$a_3 \leftrightarrow (\neg c \wedge a_4)$	$= (\neg a_3 \vee \neg c)$	$\wedge (\neg a_3 \vee a_4)$	$\wedge (a_3 \vee c \vee \neg a_4)$
$a_4 \leftrightarrow (d \vee e)$	$= (\neg a_4 \vee d \vee e)$	$\wedge (a_4 \vee \neg d)$	$\wedge (a_4 \vee \neg e)$

CNF conversion: Tseitin's encoding

$$\phi := (a \wedge b) \vee (\neg c \wedge (d \vee e))$$

CNF(ϕ) =

$$\begin{array}{lllll} (\neg a_1 \vee a_2 \vee a_3) & \wedge & (a_1 \vee \neg a_2) & \wedge & (a_1 \vee \neg a_3) & \wedge \\ (\neg a_2 \vee a) & \wedge & (\neg a_2 \vee b) & \wedge & (a_2 \vee \neg a \vee \neg b) & \wedge \\ (\neg a_3 \vee \neg c) & \wedge & (\neg a_3 \vee a_4) & \wedge & (a_3 \vee c \vee \neg a_4) & \wedge \\ (\neg a_4 \vee d \vee e) & \wedge & (a_4 \vee \neg d) & \wedge & (a_4 \vee \neg e) & \wedge \end{array}$$

a_1

Resolution

$$\frac{(\textcolor{red}{l}, \textcolor{blue}{l}_1, \dots, \textcolor{blue}{l}_n) \quad (\neg \textcolor{red}{l}, \textcolor{green}{l}'_1, \dots, \textcolor{green}{l}'_m)}{(\textcolor{blue}{l}_1, \dots, \textcolor{blue}{l}_n, \textcolor{green}{l}'_1, \dots, \textcolor{green}{l}'_m)}$$

Examples:

- ▶ $\frac{(a \vee b) \quad (\neg a \vee c)}{(b \vee c)}$
- ▶ $\frac{(a \vee b) \quad (\neg a \vee b)}{(b)}$
- ▶ $\frac{(a \vee b) \quad (\neg a \vee \neg b)}{(\text{true})}$
- ▶ $\frac{(a) \quad (\neg a)}{()}$

Resolution: completeness

$$\begin{aligned} & (a \vee P_1) \wedge \dots \wedge (a \vee P_n) \wedge (\neg a \vee Q_1) \wedge \dots (\neg a \vee Q_m) \wedge R \\ & \quad \Leftrightarrow \\ & (P_1 \vee Q_1) \wedge \dots \wedge (P_1 \vee Q_m) \wedge \dots (P_n \vee Q_1) \wedge \dots (P_n \vee Q_m) \wedge R \end{aligned}$$

Similar: Quantifier elimination

$$\phi \Leftrightarrow \phi[\text{true}/a] \vee \phi[\text{false}/a]$$