Satisfiability Checking

The Simplex Algorithm

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Outline

Gaussian Elimination

- Given a linear system $Ax = b$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{pmatrix}$$

- Manipulate $A|b$ to obtain an upper-triangular form

$$\left(\begin{array}{cccc|c} a'_{11} & a'_{12} & \dots & a'_{1k} & b'_1 \\ 0 & a'_{22} & \dots & a'_{2k} & b'_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a'_{kk} & b'_k \end{array}\right)$$

Gaussian Elimination

Then, solve backwards from k's row according to:

$$x_i = \frac{1}{a'_{ii}}(b'_i - \sum_{j=i+1}^{k} a'_{ij}x_j)$$

Example

$$\begin{pmatrix} 1 & 2 & 1 \\ -2 & 3 & 4 \\ 4 & -1 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 9 \end{pmatrix} \Rightarrow \quad \left(\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ -2 & 3 & 4 & 3 \\ 4 & -1 & -8 & 9 \end{array}\right)$$

$$\begin{array}{rcl} R3 & = & (\ 4,\ -1,\ -8 \ |\ 9\ ) \\ -4R1 & = & (\ -4,\ -8,\ -4 \ |\ -24\ ) \\ R3 & += & -4R1 \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ -2 & 3 & 4 & 3 \\ 0 & -9 & -12 & -15 \end{array}\right)$$

$$\begin{array}{rcl} R2 & = & (\ -2,\ 3,\ 4 \ |\ 3\ ) \\ 2R1 & = & (\ 2,\ 4,\ 2 \ |\ 12\ ) \\ R2 & += & 2R1 \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & 7 & 6 & 15 \\ 0 & -9 & -12 & -15 \end{array}\right)$$

$$\begin{array}{rcl} R3 & = & (\ 0,\ -9,\ -12 \ |\ -15\ ) \\ \frac{9}{7}R2 & = & (\ 0,\ 9,\ \frac{6 \cdot 9}{7} \ |\ \frac{15 \cdot 9}{7}\ ) \\ R3 & += & \frac{9}{7}R2 \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & 7 & 6 & 15 \\ 0 & 0 & -\frac{30}{7} & \frac{30}{7} \end{array}\right)$$

Now: $x_3 = -1$, $x_2 = 3$, $x_1 = 1$. Problem solved!

Satisfiability with Simplex

- Simplex was originally designed for solving the optimization problem:

  $\max \vec{c}\,\vec{x}$
  s.t.
  $A\vec{x} \leq \vec{b}, \quad \vec{x} \geq \vec{0}$

- We are only interested in the feasibility problem
  (= satisfiability problem).

General Simplex

- We will learn a variant called general simplex.
- Very suitable for solving the satisfiability problem fast.

- The input: $A\vec{x} \leq \vec{b}$
  - A is a $m \times n$ coefficient matrix
  - The problem variables are $\vec{x} = x_1, \ldots, x_n$

- First step: convert the input to *general form*

General Form

Definition (General Form)

$$A\vec{x} = 0 \quad \text{and} \quad \bigwedge_{i=1}^{m} l_i \leq s_i \leq u_i$$

A combination of

- Linear equalities of the form $\sum_i a_i x_i = 0$
- Lower and upper bounds on variables

Transformation to General Form

- Replace $\sum_i a_i x_i \bowtie b_j$ (where $\bowtie \in \{=, \leq, \geq\}$)

  with $\sum_i a_i x_i - s_j = 0$

  and $s_j \bowtie b_j$.

- Note: no $>, <$!

- $s_1, \ldots, s_m$ are called the *additional variables*

Example 1

Convert $x + y \geq 2$!

Result:
$x + y - s_1 = 0$
$s_1 \geq 2$

It is common to keep the conjunctions implicit

Example 2

Convert

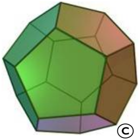
$$\begin{array}{rrl} x & +y & \geq 2 \\ 2x & -y & \geq 0 \\ -x & +2y & \geq 1 \end{array}$$

Result:

$$\begin{array}{rrrl} x & +y & -s_1 & = 0 \\ 2x & -y & -s_2 & = 0 \\ -x & +2y & -s_3 & = 0 \\ & & s_1 & \geq 2 \\ & & s_2 & \geq 0 \\ & & s_3 & \geq 1 \end{array}$$

Geometrical Interpretation

Linear inequality constraints, geometrically, define a convex polyhedron.



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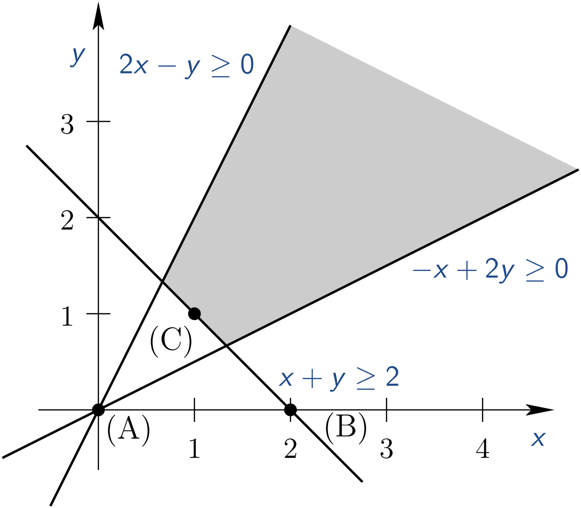
Our example from before:

$$\begin{array}{rrl} x & +y & \geq 2 \\ 2x & -y & \geq 0 \\ -x & +2y & \geq 1 \end{array}$$



Matrix Form

- Recall the general form: $A\vec{x} = 0$ and $\bigwedge_{i=1}^{m} l_i \leq s_i \leq u_i$

- A is now an $m \times (n+m)$ matrix due to the additional variables.

$$\begin{array}{rrrl} x & +y & -s_1 & = 0 \\ 2x & -y & -s_2 & = 0 \\ -x & +2y & -s_3 & = 0 \\ & & s_1 & \geq 2 \\ & & s_2 & \geq 0 \\ & & s_3 & \geq 1 \end{array}$$

$$\begin{array}{ccccc} x & y & s_1 & s_2 & s_3 \end{array}$$
$$\begin{pmatrix} 1 & 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & -1 & 0 \\ -1 & 2 & 0 & 0 & -1 \end{pmatrix}$$

The Tableau

- The diagonal part is inherent to the general form:

$$\begin{array}{ccccc} x & y & s_1 & s_2 & s_3 \end{array}$$
$$\begin{pmatrix} 1 & 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & -1 & 0 \\ -1 & 2 & 0 & 0 & -1 \end{pmatrix}$$

- Instead, we can write:

$$\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -1 & 2 \end{pmatrix} \end{array}$$

The Tableau

- The tableaux changes throughout the algorithm, but maintains its $m \times n$ structure

- Distinguish basic and nonbasic variables

$$\begin{array}{cc} & \begin{array}{cc} x & y \end{array} \leftarrow \text{Nonbasic variables} \\ \text{Basic Variables} \rightarrow \begin{array}{c} s_1 \\ s_2 \\ s_3 \end{array} & \left(\begin{array}{rr} 1 & 1 \\ 2 & -1 \\ -1 & 2 \end{array} \right) \end{array}$$

- Initially, basic variables = the additional variables

The Tableau

- Notation:

$\mathcal{B}$	the basic variables
$\mathcal{N}$	the nonbasic variables

- The tableaux is simply a different notation for the system

$$\bigwedge_{s_i \in \mathcal{B}} \left(s_i = \sum_{x_j \in \mathcal{N}} a_{ij} x_j \right)$$

- The basic variables are also called the dependent variables.

Data Structures

- Simplex maintains:
  - The tableau,
  - an assignment α to all variables,
  - an assignment to the bounds.

- Initially,
  - $\mathcal{B}$ = additional variables,
  - $\mathcal{N}$ = problem variables,
  - $\alpha(x_i) = 0$ for $i \in \{1, ..., n+m\}$

Invariants

- Two invariants are maintained throughout:
  1. $A\vec{x} = 0$
  2. All nonbasic variables satisfy their bounds
- The basic variables do not need to satisfy their bounds.
- Can you see why these invariants are maintained initially?

Invariants

- The initial assignment satisfies $A\vec{x} = 0$
- If the bounds of all basic variables are satisfied by α, return "satisfiable".
- Otherwise... *pivot*.

Pivoting

1. Find a basic variable x_i that violates its bounds.
   Suppose that $\alpha(x_i) < l_i$.

2. Find a nonbasic variable x_j such that
   - $a_{ij} > 0$ and $\alpha(x_j) < u_j$, or
   - $a_{ij} < 0$ and $\alpha(x_j) > l_j$.

   Why? Such a variable is called suitable.

3. If there is no suitable variable, return "unsatisfiable".

   Why?

Pivoting x_i and x_j (1)

1. Solve equation i for x_j:

$$\text{From:} \quad x_i = a_{ij}x_j + \sum_{k \neq j} a_{ik}x_k$$

$$\text{To:} \quad x_j = \frac{x_i}{a_{ij}} - \sum_{k \neq j} \frac{a_{ik}}{a_{ij}} x_k$$

2. Swap x_i and x_j, and update the i-th row accordingly

From:

a_{i1}	...	a_{ij}	...	a_{in}

To:

$\frac{-a_{i1}}{a_{ij}}$	...	$\frac{1}{a_{ij}}$	...	$\frac{-a_{in}}{a_{ij}}$

Pivoting x_i and x_j (2)

3. Update all other rows:
   Replace x_j with its equivalent obtained from row i:

$$x_j = \frac{x_i}{a_{ij}} - \sum_{k \neq j} \frac{a_{ik}}{a_{ij}} x_k$$

4. Update α as follows:
   - Increase $\alpha(x_j)$ by $\theta = \dfrac{l_i - \alpha(x_i)}{a_{ij}}$
     Now x_j is a basic variable: it may violate its bounds.
     Update $\alpha(x_i)$ accordingly.
     Q: What is $\alpha(x_i)$ now?
   - Update α for all other basic (dependent) variables.

Pivoting: Example (1)

- Recall the tableau and constraints in our example:

	x	y
s_1	1	1
s_2	2	-1
s_3	-1	2

$$\begin{array}{ccc} 2 & \leq & s_1 \\ 0 & \leq & s_2 \\ 1 & \leq & s_3 \end{array}$$

- Initially, α assigns 0 to all variables

  $\Longrightarrow$ Violated are the bounds of s_1 and s_3

- We will fix s_1.
- x is a *suitable* nonbasic variable for pivoting.
  It has no upper bound!
- So now we pivot s_1 with x

Pivoting: Example (2)

	x	y
s_1	1	1
s_2	2	-1
s_3	-1	2

$$\begin{array}{rcl} 2 & \leq & s_1 \\ 0 & \leq & s_2 \\ 1 & \leq & s_3 \end{array}$$

- Solve 1st row for x:

$$s_1 = x + y \iff x = s_1 - y$$

- Replace x in other rows:

$$\begin{array}{lcl} s_2 = 2(s_1 - y) - y & \iff & s_2 = 2s_1 - 3y \\ s_3 = -(s_1 - y) + 2y & \iff & s_3 = -s_1 + 3y \end{array}$$

Pivoting: Example (3)

$$\begin{array}{rcl} x & = & s_1 - y \\ s_2 & = & 2s_1 - 3y \\ s_3 & = & -s_1 + 3y \end{array}$$

This results in the following new tableau:

	s_1	y
x	1	−1
s_2	2	−3
s_3	−1	3

$$\begin{array}{rcl} 2 & \leq & s_1 \\ 0 & \leq & s_2 \\ 1 & \leq & s_3 \end{array}$$

What about the assignment?

- We should increase x by $\theta = \frac{2-0}{1} = 2$
- Hence, $\alpha(x) = 0 + 2 = 2$
- Now s_1 is equal to its lower bound: $\alpha(s_1) = 2$
- Update all the others

Pivoting: Example (4)

The new state:

	s_1	y
x	1	-1
s_2	2	-3
s_3	-1	3

$$\begin{aligned} \alpha(x) &= 2 \\ \alpha(y) &= 0 \\ \alpha(s_1) &= 2 \\ \alpha(s_2) &= 4 \\ \alpha(s_3) &= -2 \end{aligned}$$

$$\begin{aligned} 2 &\leq s_1 \\ 0 &\leq s_2 \\ 1 &\leq s_3 \end{aligned}$$

- Now s_3 violates its lower bound
- Which nonbasic variable is suitable for pivoting?
  That's right… y
- We should increase y by $\theta = \frac{1-(-2)}{3} = 1$

Pivoting: Example (5)

The final state:

	s_1	s_3
x	$2/3$	$-1/3$
s_2	1	-1
y	$1/3$	$1/3$

$$\begin{array}{rcl} \alpha(x) & = & 1 \\ \alpha(y) & = & 1 \\ \alpha(s_1) & = & 2 \\ \alpha(s_2) & = & 1 \\ \alpha(s_3) & = & 1 \end{array}$$

$$\begin{array}{rcl} 2 & \leq & s_1 \\ 0 & \leq & s_2 \\ 1 & \leq & s_3 \end{array}$$

All constraints are satisfied.

Observations I

The additional variables:

- Only additional variables have bounds.
- These bounds are permanent.
- Additional variables enter the base only on extreme points (their lower or upper bounds).
- When entering the base, they shift towards the other bound and possibly cross it (violate it).

Observations II

Q: Can it be that we pivot x_i, x_j and then pivot x_j, x_i and thus enter a (local) cycle?

A: No.

- For example, suppose that $a_{ij} > 0$.
- We increased $\alpha(x_j)$ so now $\alpha(x_i) = l_i$.
- After pivoting, possibly $\alpha(x_j) > u_j$, but $a'_{ij} = 1/a_{ij} > 0$, hence the coefficient of x_i is not suitable

Termination

Is termination guaranteed?

- Not obvious. Perhaps there are bigger cycles.

- In order to avoid circles, we use Bland's rule:
  1. Determine a total order on the variables
  2. Choose the first basic variable that violates its bounds, and the first nonbasic suitable variable for pivoting.
  3. It can be shown that this guarantees that no base is repeated, which implies termination.

General simplex With Bland's Rule

1. Transform the system into the general form

$$A\vec{x} = 0 \quad \text{and} \quad \bigwedge_{i=1}^{m} l_i \le s_i \le u_i \ .$$

2. Set $\mathcal{B}$ to be the set of additional variables $s_1, \ldots, s_m$.
3. Construct the tableau for A.
4. Determine a fixed order on the variables.
5. If there is no basic variable that violates its bounds, return "satisfiable". Otherwise, let x_i be the first basic variable in the order that violates its bounds.
6. Search for the first suitable nonbasic variable x_j in the order for pivoting with x_i. If there is no such variable, return "unsatisfiable".
7. Perform the pivot operation on x_i and x_j.
8. Go to step 5.