

Satisfiability Checking

The Omega Test

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Theory of Hybrid Systems
Informatik 2

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The Omega Test

- Goal: Decide satisfiability for conjunctions of **linear** constraints of the form

$$\sum_{0 \leq i \leq n} a_i x_i \geq 0$$

over **integers**.

- Original application:
Program optimizations done by a compiler.
- Extension of *Fourier-Motzkin* variable elimination:
 - Pick one variable and eliminate it
 - Continue until all variables but one are eliminated

Preprocessing (1)

Normalize coefficients: divide by the GCD

$$8x + 6y \leq 0 \longrightarrow 4x + 3y \leq 0$$

$$4y \geq 1 \longrightarrow y \geq \lceil 1/4 \rceil$$

$$3x + 3y = 2 \longrightarrow x + y = 2/3 \longrightarrow \text{UNSAT}$$

'Tightening'

Eliminate equalities

- Assume an equation $\sum_{0 \leq i \leq n} a_i x_i = 0$.
- If $a_k = 1$ for some k then use this equation to eliminate x_k by the substitution $[-\sum_{0 \leq i \leq n, i \neq k} a_i x_i / x_k]$.
- Otherwise, pick variable x_k and make a_k positive
- Define $\widehat{a \bmod b} := a - b \lfloor a/b + 1/2 \rfloor$
- Let $m = a_k + 1$
- Note that $\widehat{a_k \bmod m} = -1$

Eliminate equalities

- Create a **new variable** σ and add:

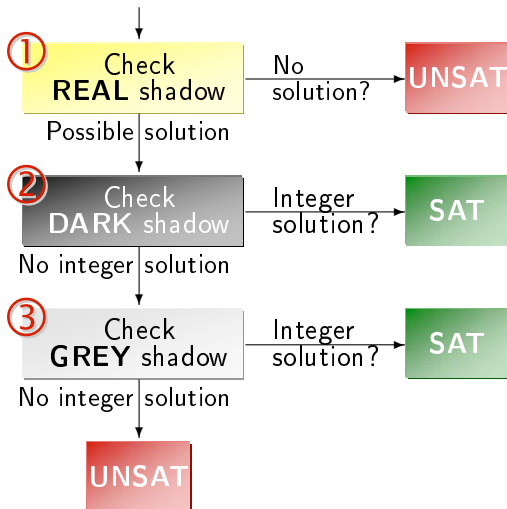
$$\sum_i (a_i \widehat{\bmod} m) x_i = m\sigma + b \widehat{\bmod} m$$

- Solve for x_k :

$$x_k = -m\sigma - b \widehat{\bmod} m + \sum_{i \neq k} (a_i \widehat{\bmod} m) x_i$$

- **Q:** What is the point of adding a constraint to eliminate one?

Overview of the Omega Test



①

Check
REAL shadow

- Assume we eliminate variable z
- For each pair of upper/lower bound:

$$\begin{array}{ll} \beta & \leq bz \\ c\beta & \leq cbz \end{array} \quad \begin{array}{ll} cz & \leq \gamma \\ cbz & \leq b\gamma \end{array} \quad (b, c > 0)$$

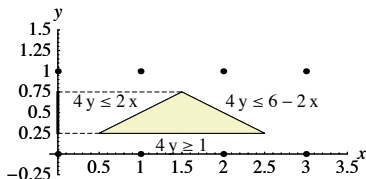
- Constraint for real shadow:

$$c\beta \leq b\gamma$$

The real shadow: Example 1

①

Check
REAL shadow



$$4y \leq 2x$$

$$4y \leq -2x + 6$$

$$4y \geq 1$$

Eliminate x :

$$4y \leq 2x$$

$$4y \leq -2x + 6$$

$$4y \leq 6 - 4y$$

$$8y \leq 6$$

Real Shadow:

$$8y \leq 6 \implies y \leq 0.75$$

$$4y \geq 1 \implies y \geq 0.25$$

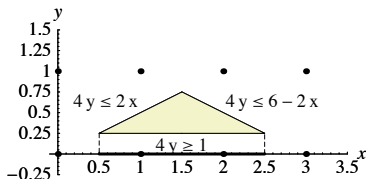
No integer solution

$\implies$ Original problem
has no solution

The real shadow: Example II

①

Check
REAL shadow



$$4y \leq 2x$$

$$4y \leq -2x + 6$$

$$4y \geq 1$$

Let's eliminate y instead:

$$1 \leq 4y$$

$$4y \leq 2x$$

$$1 \leq 2x$$

$$1 \leq 4y$$

$$4y \leq -2x + 6$$

$$1 \leq -2x + 6$$

Real Shadow:

$$1 \leq 2x$$

$$1 \leq -2x + 6$$

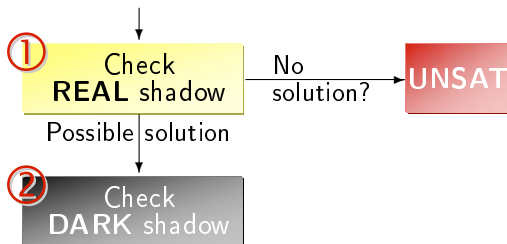
$$x \leq 0.5$$

$$x \geq 2.5$$

Integer solution!

But original problem
has no integer solution!

From Real to Dark Shadow



- An integer solution for the REAL shadow **does not guarantee** that there is an integer solution for the original problem
- Thus, we check the **DARK shadow** next

Idea of the dark shadow

2

Check
DARK shadow

- Idea of the DARK shadow:

$$\begin{array}{lcl} \beta & \leq & bz \quad | : b \\ \frac{\beta}{b} & \leq & z \end{array} \qquad \begin{array}{lcl} cz & \leq & \gamma \quad | : c \\ z & \leq & \frac{\gamma}{c} \end{array} \qquad z \in \mathbb{N}$$

- How to compute the dark shadow?
- Try to *prove* that there is an integer z between $\frac{\beta}{b}$ and $\frac{\gamma}{c}$

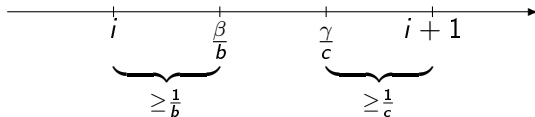
Dark shadow: Proof by contradiction

2

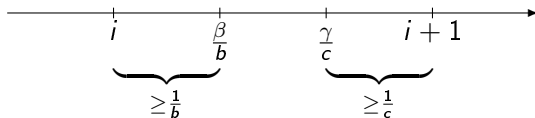
Check
DARK shadow

Assume there is no integer z between $\frac{\beta}{b}$ and $\frac{\gamma}{c}$. Then:

Let $i := \lfloor \frac{\beta}{b} \rfloor \quad i \in \mathbb{Z}$



Dark shadow: Proof by contradiction



$$\frac{\beta}{b} - i \geq \frac{1}{b}$$

$$i + 1 - \frac{\gamma}{c} \geq \frac{1}{c}$$

$$\frac{\beta}{b} + 1 - \frac{\gamma}{c} \geq \frac{1}{b} + \frac{1}{c} \quad | \cdot c \cdot b$$

$$c\beta + cb - b\gamma \geq c + b \quad | - cb$$

$$c\beta - b\gamma \geq -cb + c + b \quad | \cdot (-1)$$

$$b\gamma - c\beta \leq cb - c - b$$

Dark shadow: Proof by contradiction

- From previous slide:

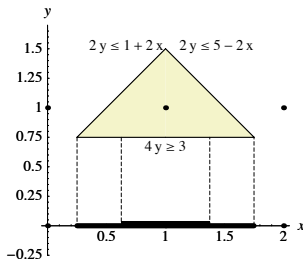
$$\begin{aligned} & b\gamma - c\beta \leq cb - c - b \\ \iff & \neg(b\gamma - c\beta > cb - c - b) \\ \iff & \neg(b\gamma - c\beta \geq cb - c - b + 1) \\ \iff & \neg(b\gamma - c\beta \geq \underbrace{(c-1)(b-1)}_*) \end{aligned}$$

- Thus, if * holds, we know that there must be an integer solution.
- If $c = 1$ or $b = 1$, then this is the same as the real shadow.
This case is called an **exact projection**.

Example for the dark shadow

2

Check
DARK shadow



$$2y \leq 2x + 1$$

$$2y \leq -2x + 5$$

$$4y \geq 3$$

Eliminate y with the dark shadow:

$$2y \leq 2x + 1$$

$$4y \geq 3$$

$$4(2x + 1) - 2 \cdot 3 \geq (2 - 1)(4 - 1)$$

$$4y \geq 3$$

$$2y \leq -2x + 5$$

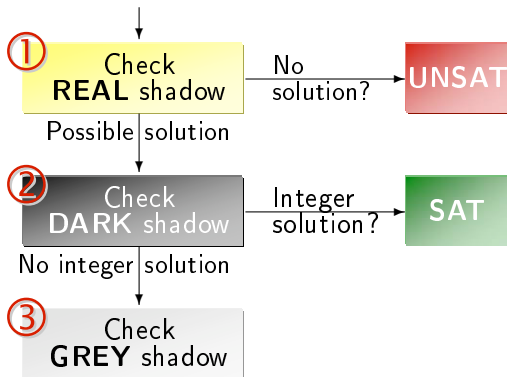
$$4(-2x + 5) - 2 \cdot 3 \geq (2 - 1)(4 - 1)$$

Dark Shadow:

$$\begin{array}{l} x \geq 5/8 \\ x \leq 11/8 \end{array}$$

$\Rightarrow$ Integer solution!

From Dark to Grey Shadow



- No integer solution in the DARK shadow **does not guarantee** that there is no integer solution for the original problem
- Thus, we check the **GREY shadow** next

The grey shadow

3

Check
GREY shadow

Idea of the Grey shadow

If the real shadow R has integer solutions,
but the dark shadow D does not, search $R \setminus D$.

$$\begin{array}{ll} \text{In } R: & b\gamma \geq cbz \geq c\beta \\ \text{Not in } D: & cb - c - b \geq b\gamma - c\beta \\ \iff & cb - c - b + c\beta \geq b\gamma \\ \Rightarrow & cb - c - b + c\beta \geq cbz \geq c\beta \quad | : c \\ & (cb - c - b)/c + \beta \geq bz \geq \beta \end{array}$$

The grey shadow

③

Check
GREY shadow

- Try all values of z such that

$$(cb - c - b)/c + \beta \geq bz \geq \beta$$

- Optimization: find the largest coefficient c in any upper bound and try the following for each lower bound $bz \geq \beta$:

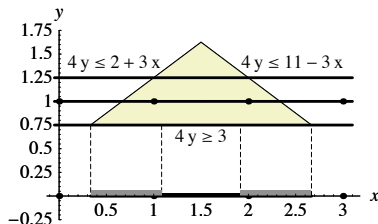
$$bz = \beta + i \quad \text{for } (cb - c - b)/c \geq i \geq 0$$

- As before, combine this with the original problem, and solve recursively.

Example of the grey shadow

③

Check
GREY shadow



$$4y \leq 3x + 2$$

$$4y \leq -3x + 11$$

$$4y \geq 3$$

- Eliminate y :
 $c = 4, b = 4, \beta = 3$

- New constraint:
 $4y = 3 + i$ for
 $2 \geq i \geq 0$:

$$4y = 3$$

$$4y = 4$$

$$4y = 5$$

$\Rightarrow$ Integer solution
with $4y = 4$