Satisfiability Checking

Bounded Model Checking

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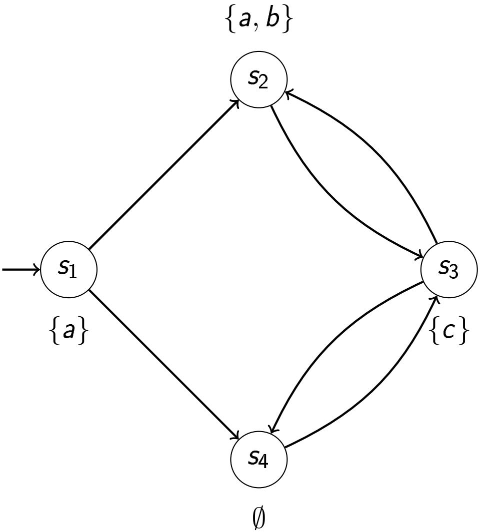
Theory of Hybrid Systems
Informatik 2

WS 11/12

Kripke structure



Kripke structure: Syntax

Definition

Let AP be a finite set of atomic propositions. A Kripke structure is a tuple $M = (S, s_{\text{init}}, T, L)$ with

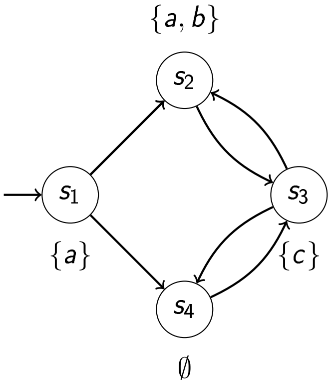
- S a finite set of states,
- $s_{\text{init}} \in S$ an initial state,
- $T \subseteq S \times S$ a transition relation,
- $L : S \rightarrow 2^{AP}$ a labeling function
  (2^{AP} denotes the powerset over AP).

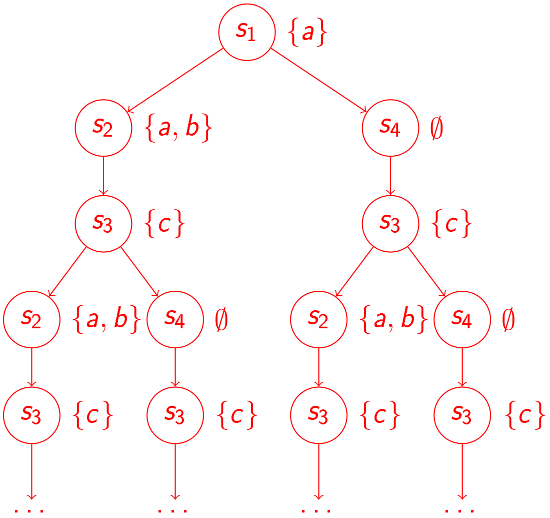
The labeling function attaches information to the system: for a state $s \in S$ the set $L(s)$ consists of those atomic propositions that hold in s.

Kripke structure: Semantics

- An (infinite) path $\pi = s_0 s_1 s_2 \dots$ of a Kripke structure $M = (S, s_{\text{init}}, T, L)$ is a sequence of states such that
  - $s_0 = s_{\text{init}}$ and
  - for all $i \geq 0$, $(s_i, s_{i+1}) \in T$.
- The behaviour of M is given by the set of all of its infinite paths.
- A finite path of M is a finite prefix of an infinite path of M.
- For a finite path $\pi = s_0 \dots s_k$ we define $|\pi| = k$.
- We write $\pi(j)$ for the jth state (starting with 0) of the path π.
- By π_j we denote the postfix of π starting at $\pi(j)$.

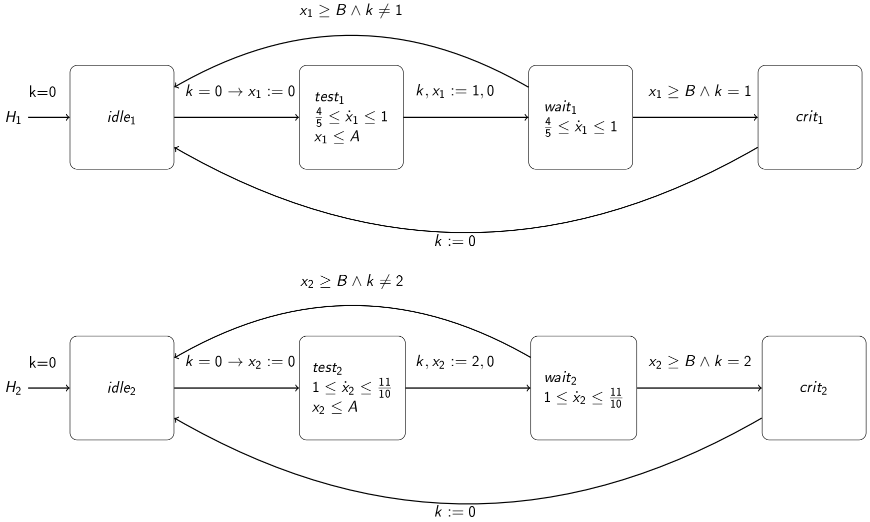
Kripke structure: Semantics





Fischer's mutual exclusion protocol

There are also more complex systems we want to deal with later.



LTL syntax

Syntax of the Linear-Time Temporal Logic (LTL):

$$\varphi ::= a \mid \varphi \land \varphi \mid \neg\varphi \mid \mathcal{X}\varphi \mid \varphi\,\mathcal{U}\,\varphi \mid \mathcal{F}\varphi \mid \mathcal{G}\varphi$$

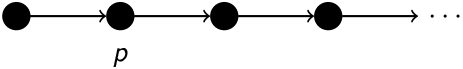
- $a \in AP$: atomic proposition
- $\mathcal{X}$: next time operator
- $\mathcal{U}$: until operator

Syntactic sugar:

- $\vee, \rightarrow, \leftrightarrow, \ldots$
- $\mathcal{F}$: finally (eventually) operator ($\mathcal{F}\varphi := \mathit{true}\,\mathcal{U}\varphi$)
- $\mathcal{G}$: globally (always) operator ($\mathcal{G}\varphi := \neg(\mathit{true}\,\mathcal{U}\neg\varphi)$)

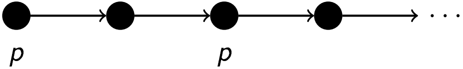
LTL semantics - Next

π:



$\pi \models \mathcal{X} p$

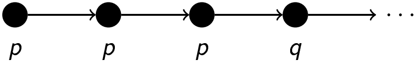
π:



$\pi \not\models \mathcal{X} p$

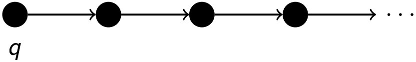
LTL semantics - Until

π:



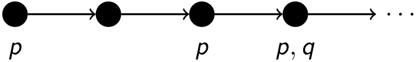
$\pi \models p\mathcal{U}q$

π:



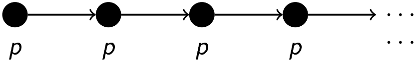
$\pi \models p\mathcal{U}q$

π:



$\pi \not\models p\mathcal{U}q$

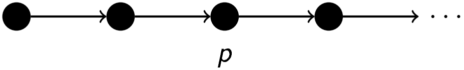
π:



$\pi \not\models p\mathcal{U}q$

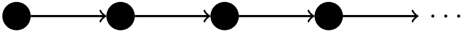
LTL semantics - Eventually

π:



$\pi \models \mathcal{F} p$

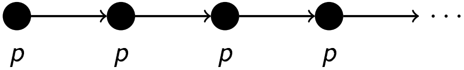
π:



$\pi \not\models \mathcal{F} p$

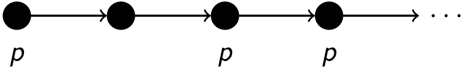
LTL semantics - Always

π:



$\pi \models \mathcal{G} p$

π:



$\pi \not\models \mathcal{G} p$

LTL semantics

Definition (LTL Semantics)

$$
\begin{array}{lll}
\pi \models p & \text{iff} & p \in L(\pi(0)) \\
\pi \models \varphi_1 \wedge \varphi_2 & \text{iff} & \pi \models \varphi_1 \text{ and } \pi \models \varphi_2 \\
\pi \models \neg\varphi & \text{iff} & \pi \not\models \varphi \\
\pi \models \mathcal{X}\varphi & \text{iff} & \pi_1 \models \varphi \\
\pi \models \varphi_1 \,\mathcal{U}\, \varphi_2 & \text{iff} & \pi_i \models \varphi_2 \text{ for some } i \geq 0 \text{ and} \\
 & & \pi_j \models \varphi_1 \text{ for all } 0 \leq j < i \\
\pi \models \mathcal{F}\varphi & \text{iff} & \pi_i \models \varphi \text{ for some } i \geq 0 \\
\pi \models \mathcal{G}\varphi & \text{iff} & \pi_i \models \varphi \text{ for all } i \geq 0
\end{array}
$$

LTL semantics

$M \models \mathbf{A}\varphi$

If all infinite paths of a Kripke structure M satisfy a property φ, then we say that the property holds for M.

$M \models \mathbf{E}\neg\varphi$

If there is an infinite path of a Kripke structure M that does not satisfy a property φ, then we say that M violates the property φ.

$M \models_k \mathbf{E}\neg\varphi$

Also finite paths can violate a property, if they contain enough information to assure the existence of an infinite path violating the property.

Model checking

Early 1980s: First implementations of Model Checking as verification technique

- Explicit representations of the transition graphs
- Problem: Due to the state explosion not applicable for most industrial settings

1990: Symbolic Model Checking

- BDDs represent *characteristic functions* of state sets
- Problem: Building the BDD may be expensive

1999: Bounded Model Checking *[Biere et al.]*

- Check the existence of finite paths of incremental length by a SAT-solver
- Problem: Incomplete (in general)

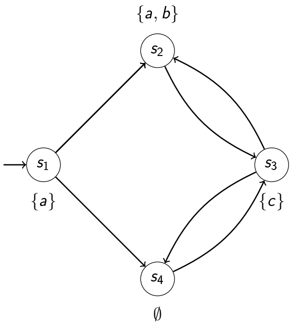
Model checking and counterexamples

- Given a Kripke structure M and an LTL property φ, a counterexample is a path of M violating φ.
- If a system is buggy, counterexamples are extremely important for detecting and fixing the error.
- Bounded model checking (BMC) is a technique to search for finite counterexamples, not only for Kripke structures, but also for more complex systems.

Finite and infinite counterexamples

Property: $\mathcal{G}\mathcal{F}a$

Negation: $\neg\mathcal{G}\mathcal{F}a = \mathcal{F}\mathcal{G}\neg a$



Infinite counterexample: $s_1\ s_4\ s_3\ s_4\ s_3\ s_4\ s_3\ \ldots$

Finite counterexample: $s_1\ \boxed{s_4}\ s_3\ \boxed{s_4}$

"Loop detected"

Definition of $\models_k$

Satisfaction relation for finite paths

$\pi \models_k^i \varphi$: the finite segment of π consisting of its ith to kth states satisfies φ

$\pi \models_k \varphi$: $\pi \models_k^0 \varphi$

Definition of $\models_k$

Satisfaction relation for finite paths with a loop

Definition

For $l \leq k$ we call an infinite path π a (k, l)-loop iff $T(\pi(k), \pi(l))$ and $\pi = u \cdot v^\omega$ with $u = \pi(0) \ldots \pi(l-1)$ and $v = \pi(l) \ldots \pi(k)$.

We call π a k-loop iff π is a (k, l)-loop for some $0 \leq l \leq k$.

Definition (Bounded semantics for a loop)

Let $k \geq 0$ and let π be a k-loop. Then an LTL formula φ is valid along π with bound k ($\pi \models_k \varphi$) iff $\pi \models \varphi$.

Definition of $\models_k$

Definition (Bounded semantics without a loop)

Let $k \geq 0$ and let π be path that is not a k-loop. Then an LTL formula φ is valid along π with bound k ($\pi \models_k \varphi$) iff $\pi \models_k^0 \varphi$, where

$$
\begin{array}{lcl}
\pi \models_k^i a & : & a \in L(\pi(i)) \\
\pi \models_k^i \neg a & : & a \notin L(\pi(i)) \\
\pi \models_k^i \varphi_1 \wedge \varphi_2 & : & \pi \models_k^i \varphi_1 \text{ and } \pi \models_k^i \varphi_2 \\
\pi \models_k^i \mathcal{X}\varphi & : & i < k \text{ and } \pi \models_k^{i+1} \varphi \\
\pi \models_k^i \varphi_1 \,\mathcal{U}\, \varphi_2 & : & \pi \models_k^j \varphi_2 \text{ for some } i \leq j \leq k \text{ and} \\
 & & \pi \models_k^n \varphi_1 \text{ for all } i \leq n < j \\
\pi \models_k^i \mathcal{F}\varphi & : & \pi \models_k^j \varphi \text{ for some } i \leq j \leq k \\
\pi \models_k^i \mathcal{G}\varphi & : & \text{false}
\end{array}
$$

Properties of $\models_k$

Lemma

Let φ be an LTL formula and let π be a path. Then

$$\pi \models_k \varphi \quad \Rightarrow \quad \pi \models \varphi.$$

Lemma

Let φ be an LTL formula and let M be a Kripke structure. Then

$$M \models \mathbf{E}\varphi \quad \Rightarrow \quad \exists k \geq 0.\ M \models_k \mathbf{E}\varphi.$$

Bounded model checking

Overview:

- Construction of a Boolean formula φ describing a finite path
  - through the underlying system
  - of length k, starting with 0,
  - and reaching a certain state of interest, i.e., violating a property.
- A SAT-solver searches for a satisfying assignment of φ
- If SAT, the resulting assignment describes a counterexample
- If UNSAT, k is incremented and the procedure starts again

Bounded model checking

Counterexamples of length k for a Kripke structure M and an LTL formula φ can be described by

$$[\![M,\varphi]\!]_k = I(s_0) \wedge T(s_0,s_1) \wedge \ldots \wedge T(s_{k-1},s_k) \wedge \neg \text{Prop}(s_0,\ldots,s_k)$$

$[\![M,\varphi]\!]_k$ is satisfiable $\iff$ there exists a finite counterexample of length k

$\rightarrow$ check $[\![M,\varphi]\!]_k$ incrementally for $k = 0, 1, \ldots$ using a suitable solver

Formula encoding

$$[\![M,\varphi]\!]_k = I(s_0) \wedge T(s_0,s_1) \wedge \ldots \wedge T(s_{k-1},s_k) \wedge \neg \mathrm{Prop}(s_0 \ldots s_k)$$

How to build this formula?

- I and T are (nearly) straightforward for Kripke structures. We build a sub-formula describing initial paths of length k:

$$[\![M]\!]_k := I(s_0) \wedge \bigwedge_{i=0}^{k-1} T(s_i, s_{i+1})$$

- This formula is called the unfolding of the transition relation

Formula encoding

$$[\![M,\varphi]\!]_k = I(s_0) \wedge T(s_0,s_1) \wedge \ldots \wedge T(s_{k-1},s_k) \wedge \neg\mathrm{Prop}(s_0 \ldots s_k)$$

How to build this formula?

- To get a counterexample for an LTL formula φ we have to find a witness for $\neg\varphi$.
- This will be encoded for paths of length k within the formula $\neg\mathrm{Prop}(s_0,\ldots,s_k)$
- The translation of the formula depends on the fact whether the considered path has a loop or not.

Encoding of loops

Loop condition: Is there a transition from s_k to a previous state?

Loop successor: Successor state of a state inside a loop

Definition

The loop condition L_k is true iff there exists a back loop from state s_k to a previous state or to itself: $L_k := \bigvee_{l=0}^{k} T(s_k, s_l)$

Definition

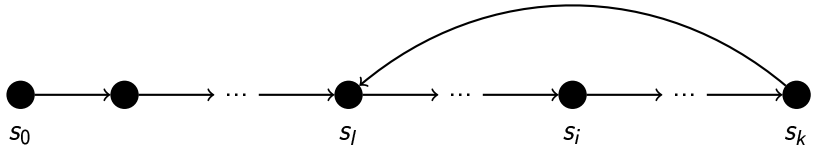
Let k, l and i be non-negative integers with $l, i \leq k$.

- $\text{succ}(i) := i + 1$, if i is inside a (k, l)-loop, i.e. $i < k$
- $\text{succ}(i) := l$ for $i = k$

Encoding of loops - Always

- Given: LTL formula φ and path π with (k, l)-loop
- Recursive translation over the sub-terms of φ and states in π
- Introduce intermediate formula of the form ${}_l[\![\cdot]\!]_k^i$
  - l start-state of the loop
  - k bound
  - i current position
- Translation rule for $\mathcal{G}\varphi$:

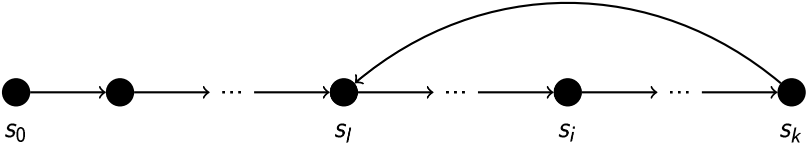
$${}_l[\![\mathcal{G}\varphi]\!]_k^i := {}_l[\![\varphi]\!]_k^i \wedge {}_l[\![\mathcal{G}\varphi]\!]_k^{\mathrm{succ}(i)}$$



Encoding of loops - Eventually

- Given: LTL formula φ and path π with (k, l)-loop
- Recursive translation over the sub-terms of φ and states in π
- Introduce intermediate formula of the form ${}_l[\![\cdot]\!]_k^i$
  - l start-state of the loop
  - k bound
  - i current position
- Translation rule for $\mathcal{F}\varphi$:

$$ {}_l[\![\mathcal{F}\varphi]\!]_k^i := {}_l[\![\varphi]\!]_k^i \vee {}_l[\![\mathcal{F}\varphi]\!]_k^{\mathrm{succ}(i)} $$



Encoding of loops

$$
\begin{aligned}
{}_l[\![p]\!]_k^i &:= p(s_i)\\
{}_l[\![\neg p]\!]_k^i &:= \neg p(s_i)\\
{}_l[\![\varphi \vee \psi]\!]_k^i &:= {}_l[\![\varphi]\!]_k^i \vee {}_l[\![\psi]\!]_k^i\\
{}_l[\![\varphi \wedge \psi]\!]_k^i &:= {}_l[\![\varphi]\!]_k^i \wedge {}_l[\![\psi]\!]_k^i\\
{}_l[\![\mathcal{G}\varphi]\!]_k^i &:= {}_l[\![\varphi]\!]_k^i \wedge {}_l[\![\mathcal{G}\varphi]\!]_k^{\mathsf{succ}(i)}\\
{}_l[\![\mathcal{F}\varphi]\!]_k^i &:= {}_l[\![\varphi]\!]_k^i \vee {}_l[\![\mathcal{F}\varphi]\!]_k^{\mathsf{succ}(i)}\\
{}_l[\![\varphi\mathcal{U}\psi]\!]_k^i &:= {}_l[\![\psi]\!]_k^i \vee \left({}_l[\![\varphi]\!]_k^i \wedge {}_l[\![\varphi\mathcal{U}\psi]\!]_k^{\mathsf{succ}(i)}\right)\\
{}_l[\![\mathcal{X}\varphi]\!]_k^i &:= {}_l[\![\varphi]\!]_k^{\mathsf{succ}(i)}
\end{aligned}
$$

Encoding without loops - Always

- Given: LTL formula φ and path π without (k, l)-loop
- Special case of loop translation
- Extension to infinite path with considering all properties beyond s_k as false
  - k bound
  - i current position
- Translation rule for $\mathcal{G}\varphi,\ i \leq k$:

$$[\![\mathcal{G}\varphi]\!]_k^i := [\![\varphi]\!]_k^i \vee [\![\mathcal{G}\varphi]\!]_k^{\mathsf{succ}(i)}$$

Encoding without loops - Eventually

- Given: LTL formula φ and path π without (k, l)-loop
- Special case of loop translation
- Extension to infinite path with considering all properties beyond s_k as false
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- Translation rule for $\mathcal{F}\varphi,\ i \leq k$:

$$[\![\mathcal{F}\varphi]\!]_k^i := [\![\varphi]\!]_k^i \vee [\![\mathcal{F}\varphi]\!]_k^{\mathsf{succ}(i)}$$

Encoding without loops

$$
\begin{aligned}
[\![p]\!]_k^i &:= p(s_i) \\
[\![\neg p]\!]_k^i &:= \neg p(s_i) \\
[\![\varphi \vee \psi]\!]_k^i &:= [\![\varphi]\!]_k^i \vee [\![\psi]\!]_k^i \\
[\![\varphi \wedge \psi]\!]_k^i &:= [\![\varphi]\!]_k^i \wedge [\![\psi]\!]_k^i \\
[\![\mathcal{G}\varphi]\!]_k^i &:= [\![\varphi]\!]_k^i \wedge [\![\mathcal{G}\varphi]\!]_k^{i+1} \\
[\![\mathcal{F}\varphi]\!]_k^i &:= [\![\varphi]\!]_k^i \vee [\![\mathcal{F}\varphi]\!]_k^{i+1} \\
[\![\varphi\mathcal{U}\psi]\!]_k^i &:= [\![\psi]\!]_k^i \vee ([\![\varphi]\!]_k^i \wedge [\![\varphi\mathcal{U}\psi]\!]_k^{i+1}) \\
[\![\mathcal{X}\varphi]\!]_k^i &:= [\![\varphi]\!]_k^{i+1}
\end{aligned}
$$

General translation to SAT

- Combining the components, BMC is encoded in propositional logic
- Given: LTL formula φ, Kripke structure M, bound k

$$[\![M,\varphi]\!]_k \;:=\; [\![M]\!]_k \wedge \Big((\neg L_k \wedge [\![\varphi]\!]_k^0) \;\vee\; \bigvee_{l=0}^{k} (T(s_k, s_l) \wedge {}_l[\![\varphi]\!]_k^0) \Big)$$

  - Unfolding of the transition relation
  - There is no back loop $\rightsquigarrow$ Translation without loops
  - All possible starting points of a loop are considered $\rightsquigarrow$ Translation for (k,l)-loop together with loop condition

Theorem

$[\![M,\varphi]\!]_k$ is satisfiable iff $M \models_k \mathbf{E}\varphi$

BMC is not complete

- Application: Start with $k = 0$ and increment until witness is found
- Termination is guaranteed iff witness exists ($M \models \mathbf{E}\varphi$)
- If no witness exists, procedure does not terminate ($M \not\models \mathbf{E}\varphi$)
- Upper bound for k to ensure property: Completeness threshold

Completeness threshold

- For each (finite state) system M, property p and given translation scheme there exists a number $\mathcal{CT}$, called completeness threshold.
- Considering $\mathcal{G}\varphi$ formulas, $\mathcal{CT}$ is equal to the reachability diameter, i.e., the minimal distance required to reach all (reachable) states of the system.

Definition (Reachability Diameter)

$$rd(M) := min\Big\{ i \mid \forall n > i.\ \forall s_0, \ldots, s_n.\ \exists t \leq i.\ \exists s'_0, \ldots, s'_t.$$

$$\left(I(s_0) \wedge \bigwedge_{j=0}^{n-1} T(s_j, s_{j+1}) \right) \rightarrow \left(I(s'_0) \wedge \bigwedge_{j=0}^{t-1} T(s'_j, s'_{j+1}) \wedge s'_t = s_n \right) \Big\}$$

"Every state that is reachable in n steps, is also reachable in i steps."

- This yields maximal shortest paths in the system.

Completeness threshold

- Problem: One has to choose n
- Let V be the set of variables defining the states. Worst case: $n = 2^{|V|}$
- Better: Choose $n = i + 1$.

Definition (Reachability Diameter)

$$rd(M) := min\Big\{ i \mid \forall s_0, \ldots, s_{i+1}.\ \exists s'_0, \ldots, s'_i.$$

$$\left(I(s_0) \wedge \bigwedge_{j=0}^{i} T(s_j, s_{j+1}) \right) \rightarrow \left(I(s'_0) \wedge \bigwedge_{j=0}^{i-1} T(s'_j, s'_{j+1}) \wedge \bigvee_{j=0}^{i} s'_j = s_{i+1} \right) \Big\}$$

"Every state that is reachable in $i + 1$ steps, it is also reachable in i steps."

Completeness threshold

- Problem: Formula contains alternation of quantifiers
- Solution: Over-approximation of $rd(M)$

Definition (Recurrence Diameter)

$rdr(M) :=$

$$max\left\{ i \mid \exists s_0 \ldots s_i : I(s_0) \wedge \bigwedge_{j=0}^{i-1} T(s_j, s_{j+1}) \wedge \bigwedge_{j=0}^{i-1} \bigwedge_{k=j+1}^{i} s_j \neq s_k \right\}$$

"Longest loop-free initial path in M."

- As every shortest path is a loop-free path, this is an over-approximation of $rd(M)$.