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Informatik 2: Prof. Dr. Ir. Joost-Pieter Katoen
RWTH AACHEN

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Lehrstuhl für Informatik 2
Software Modeling and Verification
Prof. Dr. Ir. Joost-Pieter Katoen

Probabilistic Backward Bisimulation on interactive Markov chains

Lisa Lena von Büttner

Matrikelnummer 273838

1. Gutachter: Prof. Dr. Ir. Joost-Pieter Katoen

2. Gutachter: Prof. Dr. Thomas Noll

Betreuer: M. Sc. Arpit Sharma

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(Lisa Lena von Büttner)

Abstract In the year 2010, Hermanns and Katoen [13] defined and published the *probabilistic strong forward bisimulation* on Interactive Markov Chains (IMCs), which allows a reduction of the system's state space while, for instance, preserving time-bounded reachability probabilities. Since realistic systems consist of millions of states, it is of crucial importance to algorithmically and efficiently reduce their state spaces without too much loss of information provided by the original model, such that a more efficient verification in time and space can be performed on the minimized system. Great effort has already been put in the analysis of the probabilistic forward bisimulation and its property preservation abilities [11, 13, 23] and to the best of our knowledge, a probabilistic backward variant definition and analysis on particular properties (specified in Continuous Stochastic Logic) solely exists for Continuous-Time Markov Chains (CTMCs) [21], which do not exhibit nondeterministic behavior. Thus, the central challenge of this Master's thesis focusses on the definition of a *probabilistic strong backward bisimulation* relation on IMCs and to give proofs or disproofs for the preservation of three probabilistic timed properties, i.e. minimum and maximum timed reachability, expected-time and long-run average.

Zusammenfassung Die von Hermanns und Katoen [13] im Jahre 2010 definierte und publizierte *probabilistische starke vorwärts Bisimulation* auf Interactive Markov Chains (IMCs) erlaubt eine Reduktion des Zustandsraumes des Systems und erhält dabei wichtige Informationen wie beispielsweise zeitlich-begrenzte Erreichbarkeitswahrscheinlichkeiten. Da realistische Systeme aus millionen Zuständen bestehen, ist es von äußerster Wichtigkeit den Zustandsraum algorithmisch und effizient minimieren zu können, ohne dabei die zu untersuchenden Eigenschaften des Eingangsmodells zu kontaminieren, sodass mit effizientem Zeit- und Platzaufwand das minimierte System verifiziert werden kann. Die probabilistische vorwärts Bisimulation wurde intensiv erforscht, insbesondere im Hinblick auf die Erhaltung diverser Systemmerkmale [11, 13, 23] und uns ist bezüglich der Rückwärtsvariante lediglich die Definition und die Analyse auf bestimmte Eigenschaften (spezifiziert in Formeln der Continuous Stochastic Logic) für Continuous-Time Markov Chains (CTMCs) bekannt [21], die jedoch kein nichtdeterministisches Verhalten aufweisen. Die zentrale Aufgabe dieser Masterarbeit konzentriert sich daher auf eine Definition der *probabilistischen rückwärts Bisimulation* auf IMCs, sowie das Beweisen oder Widerlegen der Erhaltung dreier probabilistischer zeitabhängiger Systemeigenschaften, die minimale und maximale zeitabhängige Erreichbarkeit, erwartete Zeit und langfristige durchschnittliche Zeit.

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Contents

1. Introduction

In 2002, Hermanns published a book [12] where he introduced the model of Interactive Markov Chains (IMC), which, on the one hand, extends conventional Labeled Transition Systems (LTS) by exponentially distributed delays, i.e. next to action transitions there is a second type called *Markovian transitions* of the form $s \stackrel{\lambda}{\Longrightarrow} t$ with $\lambda \in \mathbb{R}_{>0}$ being the parameter of the negative exponential distribution that conducts the random delay. On the other hand, it allows for nondeterminism and compositional modelling [23]. Omitting Markovian transitions leads back to classical LTSs, whereas dropping action transitions results in Continuous Time Markov Chains (CTMC), which represent "the simplest but also most widespread class of performance and dependability models" [13]. However, they lack of the support of nondeterminism and compositional modelling [23] but there exists a variety of performance and dependability analysis tools (e.g. [1], [14], [17], etc.).
Markov Chains form the underlying semantics of formalisms like stochastic Petri nets [3] and stochastic process algebras [12], [15], which are used to model systems with nontrivial probabilistic behavior [21]. An Interactive Markov Chain is one of these semantical models and Zhang and Neuhäußer [23] and Katoen et al. [16] developed algorithms for analyzing and minimizing these systems, as it is of crucial importance to keep them as small as possible if analyses are to be performed in an acceptable amount of time. In 2010, Hermanns and Katoen [13] published a paper that defined a probabilistic bisimulation relation on IMCs that groups its states into bisimulation-equivalent sets and thus, can be used for state space reductions of IMC models. Due to the compositional nature of IMCs and the fact that the authors' definition of bisimulation is a congruence wrt. parallel composition, huge IMCs can be minimized in a component-wise manner without the loss of (e.g.) time-bounded reachability properties. Sproston and Donatelli [21] developed a probabilistic bisimulation relation (pbb, for short) on CTMCs in 2006 that is not defined in the traditional forward manner but rather in a backward one. Roughly speaking, states s and t are identified to be bisimilar whenever they exhibit an equivalent (in the sense of the bisimulation relation) background, i.e. for every equivalence class C, if C reaches s by some rate λ, then so is t, and vice versa. It has been proven that the bisimulation quotient of a given CTMC model preserves a fragment of the Continuous Stochastic Logic (CSL) (see [19], [21], [23]), which specifies complex properties of interest referring to the performance of stochastic systems. Mainly these two innovative approaches opened a research question that, to the best of our knowledge, has not been investigated yet. The leading question is whether we can define a probabilistic backward bisimulation relation on Interactive Markov Chains that permits minimization of the system by minimizing its individual components without (too much) loss of information, such that model checking algorithms performed on the quotient return reliable and quantifiably precise solutions. Of course, since Sproston and Donatelli [21] have shown that pbb on CTMCs preserves a subclass of CSL, we cannot expect the probabilistic backward bisimulation on IMCs to preserve full CSL formulae. As a

matter of fact, in this thesis we will find out that a component-wise minimization according to the equivalence conditions this relation entails, reveals an origin-equivalent model only in very restrictive settings and does not preserve even the basic properties, such as timed reachability probability, expected time and long-run average (Definitions see [11,23]).

The thesis is organized as follows. In Chapter 2 the reader is provided with an overview on the current state of the art and with the mathematical tools and definitions used throughout this work. Chapter 3 gives a definition of a probabilistic backward bisimulation on IMCs and of the corresponding quotient system that is obtained when bisimulation is applied. We further investigate under which circumstances original and quotient system are bisimilar in the sense of our previously introduced bisimulation relation and show that pbb is a congruence wrt. parallel composition. The chapter is concluded with the exhibition of the main differences to the forward variant of Hermanns and Katoen [13]. Chapter 4 focusses on the quantitative timed analysis of original and reduced IMC and provides counterexample for three properties that are not preserved in the quotient system. Chapter 5 summarizes and concludes this thesis.

2. Preliminaries

In the following, we will give a short overview of the current state of the art concerning the analysis and model checking of Interactive Markov Chains. Afterwards, an introduction to mathematical tools and notations will be provided, which are used throughout this thesis and which are based on or taken from the approaches of [11, 13, 18–20, 23].

2.1. State of the Art

The introduction of Interactive Markov Chains in 2002 by Hermanns [12] "as an orthogonal extension of Labeled Transition Systems and Continuous Time Markov Chains [1] [...] received attention in academic as well as industrial settings [4], [6], [7]" [23]. Until 2010, however, model checking IMCs was restricted to those that could be transformed into CTMCs, for which Baier et al. [1] have proven that the CSL model checking problem for rational time-bounds is decidable and tools like PRISM [17] and $E \vdash MC^2$ [14] implement algorithms for CSL model checking. This problem was solved by Zhang and Neuhäußer [23], who presented an efficient algorithm for verifying CSL formulae on IMCs that is linear in the size of the formula to be checked and quadratic in the size of the IMC. Furthermore, they provided a fixed point characterization for IMCs to compute the maximum/minimum timed reachability probability. In the same year, Hermanns and Katoen [13] gave a formal definition of strong and weak bisimulation relations for IMCs that allowed to minimize the model at hand efficiently in a component-wise manner, while preserving, e.g., time-bounded reachability probabilities. Guck et al. [11] presented algorithms "for determining the extremal expected time of reaching a set of states, and the long-run average of time spent in a set of states". Guck also implemented the tool Interactive Markov Chain Analyzer (IMCA) [10], which investigates the system on time-bounded reachability, interval-bounded reachability, unbounded reachability, expected time, expected steps, long-run average, etc. The master's thesis by Sazonov [20] in 2013 focussed on the property preservation under a number of probabilistic forward bisimulation relations, such as strong, naive weak, weak, and branching bisimulation, on Markov Automata (MA) of which IMCs are a submodel. It turns out that the quotients under strong, naive weak, and branching bisimulation preserve minimum expected time and minimum long run average for MAs and hence, for IMCs. Without going into detail, we summarize that, for the weak case, this only holds under certain conditions like the existence of at least one Markovian state within the set of goal states for minimum expected time.

In 2006, Sproston and Donatelli introduced a backward variant of probabilistic strong bisimulation on CTMCs, a submodel of IMCs. This equivalence relation turns out to preserve a fragment of CSL, namely those formulae that are probabilistic or steady-state operator nesting-free, and the probability of a class in the quotient CTMC of satisfying a formula

within this subclass "is the average of the probabilities of satisfying the formula in the constituent states of the class" [21].
Altogether, these very positive and promising approaches in research fields closely related to that of IMCs, motivated for a definition of a probabilistic backward bisimulation on Interactive Markov Chains.

2.2. Interactive Markov Chains

An *Interactive Markov Chain* (IMC) is a Labeled Transition System, which consists of *states* connected via *action-labled transitions* for communication and synchronization purposes and/or Markovian *transitions*, which are labeled by rates of exponential distributions and thus enable to compute probabilities with regard to the system's future behavior.

Definition 2.2.1 (Interactive Markov Chain (IMC))
An *Interactive Markov Chain* is a tuple $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$ where

- S is a nonempty, finite set of states with initial state $s_0 \in S$,
- Act is a finite set of actions,
- $\longrightarrow \subseteq S \times Act \times S$ is set of *interactive* transitions,
- $\Longrightarrow \subseteq S \times \mathbb{R}_{>0} \times S$ is a set of *Markovian* transitions,
- AP is a set of atomic propositions, and
- $L : S \to 2^{AP}$ is a labeling function.

In the following, we will assume for each IMC $I = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$ that its initial state s_0 is labeled by the special atomic proposition z with $z \notin AP$, such that $z \in L(s_0) \subseteq AP \uplus \{ z \}$. This will be important in context of defining equivalence relations and classes on the given IMC(s).
An interactive transition $(s, \alpha, t) \in \longrightarrow$ is abbreviated by $s \xrightarrow{\alpha} t$, a Markovian transition $(s, \lambda, t) \in \Longrightarrow$ is abbreviated by $s \stackrel{\lambda}{\Longrightarrow} t$. The denumerable set of actions $Act = \{ \alpha, \beta, ... \}$ inherits special interactive actions, the so-called *internal interactive actions*, denoted (e.g.) τ, which model unobservable system behavior and which are not subject to any interaction. The set of internal actions is denoted by $Act_i \subseteq Act$. Let $Act(s) = \{ \alpha \in Act \mid \exists t \in S. s \xrightarrow{\alpha} t \}$ denote the set of enabled actions if $s \in IS$, and $Act(s) = \{ \perp \}$ if $s \in MS$. Next we classify states according to their outgoing transitions.

Definition 2.2.2 (Classification of States)
Let $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$ be an IMC, $IT(s)$ and $MT(s)$ be the set of interactive, resp. Markovian transitions leaving from $s \in S$. Then:

- s is *interactive* if $IT(s) \neq \emptyset$ and $MT(s) = \emptyset$
- s is *Markovian* if $IT(s) = \emptyset$ and $MT(s) \neq \emptyset$

- s is *hybrid* if $IT(s) \neq \emptyset$ and $MT(s) \neq \emptyset$
- s is *deadlock* if $IT(s) = \emptyset = MT(s)$

Let $IS \subseteq S$, $MS \subseteq S$, and $HS \subseteq S$ denote the interactive, Markovian, and hybrid states in IMC $\mathcal{I}$, respectively. If all states $s \in S$ are interactive, i.e. $IS = S$, then $\mathcal{I}$ is a *Labeled Transition System* (LTS). If all states are Markovian, i.e. $S = MS$, then $\mathcal{I}$ is a *Continuous-Time Markov Chain* (CTMC). We assume the reader to be familiar with syntax and semantics of LTSs (otherwise see Baier and Katoen [2], Chap. 2) and directly provide the semantics of Markovian transitions. A Markovian transition $s \overset{\lambda}{\Longrightarrow} t$ intuitively means that from state $s \in S$ of the IMC $\mathcal{I}$, state t can be reached in one step within d time units with probability $1 - e^{-\lambda \cdot d}$. The positive real value λ thus uniquely identifies a negative exponential distribution. Let $\mathbf{R}(s,t) = \sum\{ \lambda | \, s \overset{\lambda}{\Longrightarrow} t \}$ with $s \in MS$, be the *rate* to move from state s to state t. The *race condition* is the competition between two (or more) transitions $s \overset{\lambda_1}{\Longrightarrow} t_1$ and $s \overset{\lambda_2}{\Longrightarrow} t_2$ of state $s \in MS$, where the probability of IMC $\mathcal{I}$ to switch to a direct successor state t_1 or t_2 is positive in both cases, i.e. $\mathbf{R}(s,t_1) = \lambda_1 > 0$ and $\mathbf{R}(s,t_2) = \lambda_2 > 0$. In this case, the probability that a certain transition $s \overset{\lambda}{\Longrightarrow} t$ is taken, or, equivalently, wins the race, does not solely depend on the transition's rate but also on the rates of all other transitions emanating from s. Formally, we obtain that the probability to move from s to t within d time units is given by:

$$\frac{\mathbf{R}(s,t)}{\mathbf{E}(s)} \cdot (1 - e^{-\mathbf{E}(s) \cdot d}), \tag{2.1}$$

where $\mathbf{E}(s) = \sum_{t \in S} \mathbf{R}(s,t)$ denotes the *exit rate* of state s. Intuitively, it states that after a delay of at most d time units (second term), the IMC $\mathcal{I}$ moves probabilistically to a direct successor t with *discrete branching probability* $\mathbf{P}(s,t) = \frac{\mathbf{R}(s,t)}{\mathbf{E}(s)}$.

Note that throughout this thesis, we assume that the given IMCs are finite, finitely branching IMCs, and have no deadlock states.

Definition 2.2.3 (Maximal Progress)
In any IMC, internal interactive transitions take precedence over Markovian transitions.

Internal interactive transitions are not subject to interaction with the IMCs environment and hence, they can occur immediately and will not be delayed. In contrast to that, the probability of a Markovian transition to trigger instantaneously is zero. As a consequence, all Markovian transitions of hybrid states $s \in HS$ can be safely eliminated without altering the IMC's semantics.

Definition 2.2.4 (Closed IMC)
An IMC $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$ is *closed* if for each state $s \in S$ it holds that $IT(s) \neq \emptyset$ implies $MT(s) = \emptyset$ and $Act = Act_i$. Then, $HS = \emptyset$ and $S = MS \cup IS$ and all actions are internal.

Closed IMCs are not subject to any further synchronization with their environment and hence, all interactive transitions are assumed to be internal.

Definition 2.2.5 (Zenoness)
A closed IMC $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$ is *zeno* if it contains a strongly connected component with states $T \subseteq IS$, which is reachable from the initial state s_0.

Intuitively, an IMC $\mathcal{I}$ is zeno, if it can perform infinitely many actions in a finite amount of time. As this behavior is unrealistic and a consequence of modeling flaws, in the rest of this thesis we assume all closed IMCs to be non-zeno.

Definition 2.2.6 (Parallel Composition)
Let $\mathcal{I}_i = (S_i, Act_i, \longrightarrow_i, \Longrightarrow_i, s_{0,i}, AP, L_i)$ be two IMCs. The *parallel composition* of $\mathcal{I}_1$ and $\mathcal{I}_2$ w.r.t. set A of synchronization actions is defined by:

$$\mathcal{I}_1 \parallel_A \mathcal{I}_2 = (S_1 \times S_2, Act_1 \cup Act_2, \longrightarrow, \Longrightarrow, (s_{0,1}, s_{0,2}), AP, L),$$

where $L(\langle s_1, s_2 \rangle) = (L_1(s_1) \cup L_2(s_2)) \backslash \{ z \}$ if $s_1 \neq s_{0,1}$ or $s_2 \neq s_{0,2}$ and $L(\langle s_1, s_2 \rangle) = L_1(s_1) \cup L_2(s_2)$ otherwise. The transition relations $\longrightarrow$ and $\Longrightarrow$ are defined as the smallest relations satisfying

(1) $s_1 \xrightarrow{\alpha} s_{1'}$ and $s_2 \xrightarrow{\alpha} s_{2'}$ and $\alpha \in A, \alpha \neq \tau$ implies $(s_1, s_2) \xrightarrow{\alpha} (s_{1'}, s_{2'})$

(2) $s_1 \xrightarrow{\alpha} s_{1'}$ and $\alpha \notin A$ implies $(s_1, s_2) \xrightarrow{\alpha} (s_{1'}, s_2)$ for any $s_2 \in S_2$

(3) $s_2 \xrightarrow{\alpha} s_{2'}$ and $\alpha \notin A$ implies $(s_1, s_2) \xrightarrow{\alpha} (s_1, s_{2'})$ for any $s_1 \in S_1$

(4) $s_1 \stackrel{\lambda}{\Longrightarrow} s_{1'}$ implies $(s_1, s_2) \stackrel{\lambda}{\Longrightarrow} (s_{1'}, s_2)$ for any $s_2 \in S_2$

(5) $s_2 \stackrel{\lambda}{\Longrightarrow} s_{2'}$ implies $(s_1, s_2) \stackrel{\lambda}{\Longrightarrow} (s_1, s_{2'})$ for any $s_1 \in S_1$

Definition 2.2.7 (Hiding)
The *hiding* of IMC $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$ w.r.t. the set $A \subseteq Act$ of actions is the IMC $\mathcal{I} \backslash A = (S, Act \backslash A, \longrightarrow', \Longrightarrow, s_0, AP, L)$, where $\longrightarrow'$ is the smallest relation defined by:

(1) $s \xrightarrow{\alpha} s'$ and $\alpha \notin A$ implies $s \xrightarrow{\alpha}{}' s'$, and

(2) $s \xrightarrow{\alpha} s'$ and $\alpha \in A$ implies $s \xrightarrow{\tau}{}' s'$.

Notation 2.2.1 (Equivalence Classes, Quotient Space)
Let S be a set and $\mathcal{R}$ an equivalence on S. For $s \in S$, the set $[s]_{\mathcal{R}} = \{ s' \in S \mid (s, s') \in \mathcal{R} \}$ denotes the equivalence class of state s under $\mathcal{R}$ and is often referred to as the $\mathcal{R}$-equivalence class of state s. Note that $[s]_{\mathcal{R}} = [s']_{\mathcal{R}}$ for all $s, s' \in \mathcal{R}$. The quotient space of S under $\mathcal{R}$, denoted by $S/\mathcal{R} = \{ [s]_{\mathcal{R}} \mid s \in S \}$, is the set consisting of all $\mathcal{R}$-equivalence classes.

2.3. IMC semantics

In this section we provide the semantics of IMCs, i.e. paths, events, schedulers, and probability spaces. Most definitions are taken from [13,18–20,23] without any major or significant changes.

Definition 2.3.1 (Direct Successors and Predecessors)
Let $\mathcal{I}$ be an IMC and s a state in S.

- $Post_\alpha(s) = \{ t \in S \mid s \xrightarrow{\alpha} t \in \longrightarrow, \alpha \in Act \}$ is the set of *direct interactive successors* of s, where α can be internal.

- $Post_\tau(s) = \{ t \in S \mid s \xrightarrow{\tau} t \in \longrightarrow, \tau$ internal $\}$ is the set of *direct internal interactive successors* of s.
- $Post_\lambda(s) = \{ t \in S \mid s \overset{\lambda}{\Longrightarrow} t \in \Longrightarrow, \lambda \in \mathbb{R}_{\geq 0} \}$ is the set of *direct Markovian successors* of s.
- $Post(s) = Post_\alpha(s) \cup Post_\lambda(s)$ is the set of *direct successors* of s.

Similar definitions can be given for a set $C \subseteq S$ of states.

- $Post_\alpha(C) = \bigcup_{s \in C} Post_\alpha(s)$ is the set of *direct interactive successors* of a set C.
- $Post_\tau(C) = \bigcup_{s \in C} Post_\tau(s)$ is the set of *direct internal interactive successors* of a set C.
- $Post_\lambda(C) = \bigcup_{s \in C} Post_\lambda(s)$ is the set of *direct Markovian successors* of a set C.
- $Post(C) = Post_\alpha(C) \cup Post_\lambda(C)$ is the set of *direct successors* of a set C.

As in [23], we assume w.l.o.g. for the sake of simplicity and to ease the development of the theory that each internal action $\tau \in Act_i$ has a unique successor state, denoted $succ(\tau)$, if the IMC at hand is closed. Furthermore, for the rest of this section, assume the given IMC to be closed.

Definition 2.3.2 (Finite and Infinite Paths)
Let $\mathcal{I}$ be an IMC with a set of actions Act.

- A *finite* path π of $\mathcal{I}$ is a finite sequence

$$s_0 \xrightarrow{\sigma_0, t_0} s_1 \xrightarrow{\sigma_1, t_1} \dots \xrightarrow{\sigma_{n-1}, t_{n-1}} s_n, \ (n \geqslant 0),$$

  such that $s_i \in Post(s_{i-1})$, $\sigma_i \in Act$ or $\sigma_i = \perp$, and $t_i \in \mathbb{R}_{\geq 0}$, for all $0 < i \leqslant n$.

- An *infinite* path π of $\mathcal{I}$ is an infinite sequence

$$s_0 \xrightarrow{\sigma_0, t_0} s_1 \xrightarrow{\sigma_1, t_1} s_2 \xrightarrow{\sigma_2, t_2} \dots,$$

  such that $s_i \in Post(s_{i-1})$, $\sigma_i \in Act$ or $\sigma_i = \perp$, and $t_i \in \mathbb{R}_{\geq 0}$, for all $i > 0$.

The real-valued variable t_i reflects the sojourn time spent in some state s_i before moving to state s_{i+1}, referred to by $\delta(\pi, i) = t_i$. The length n of a path π reflects the number of states within π and is denoted $|\pi|$. Then, for every finite path π, $\pi\downarrow$ denotes the last state s_n on π. Note that since internal interactive transitions are executed instantaneously, we have that $\sigma_i = \tau$ implies $t_i = 0$. Furthermore, as the probability of a Markovian transition to occur immediately is zero, it follows that if $\sigma_i = \perp$ then the sojourn time in state s_i is $\delta(\pi, i) = t_i > 0$. The notation $\pi@t \in (S^* \cup S^\omega)$ is the sequence of states traversed on π at time point $t \in \mathbb{R}_{\geq 0}$. Note that at some time point t, path π will in general occupy several rather than a single state because of instantaneous transitions. (For more details, refer to Zhang and Neuhäußer [23])
The *prefix* of π is referred to by $\pi[0..i] = s_0 \xrightarrow{\sigma_0, t_0} s_1 \xrightarrow{\sigma_1, t_1} \dots \xrightarrow{\sigma_{i-1}, t_{i-1}} s_i$.
In the following, let $Act_\perp = Act \cup \{ \perp \}$, such that $Act_\perp(s) = Act(s)$ if $s \in IS$ and $Act_\perp(s) = \{ \perp \}$ otherwise.

Definition 2.3.3 (Set of Paths)
Let $\mathcal{I}$ be an IMC. A path in $\mathcal{I}$ is a concatenation of a state and a sequence of *combined transitions* from the set $\Omega = Act_{\perp} \times \mathbb{R}_{\geq 0} \times S$, i.e. $\pi = s_0 \circ m_0 \circ m_1 \circ ... \circ m_{n-1}$ with $m_i = (\sigma_i, t_i, s_{i+1}) \in \Omega$. The paths in $\mathcal{I}$ can be classified as follows.

- $Paths^n(\mathcal{I}) = S \times \Omega^n$ is the set of finite paths of length n
- $Paths^*(\mathcal{I}) = S \times \Omega^*$ is the set of all finite paths
- $Paths^\omega(\mathcal{I}) = S \times \Omega^\omega$ is the of all infinite paths
- $Paths(\mathcal{I}) = ((S \times \Omega^*) \cup (S \times \Omega^\omega))$ is the set of all finite and infinite paths $Paths^* \cup Paths^\omega$

The reference to $\mathcal{I}$ will be omitted if it is clear from the context.
A combined transition (σ, t, s) describes the exponentially distributed time-point t to move to a successor state s from the current state if $\sigma \in \perp$. If $\sigma \in Act$ then time point t reflects the sojourn time spent in the current state under the assumption that a scheduler (see Definition 2.3.6), which resolves nondeterminisims in the system, decided action σ to be executed in the current state. Note that the sets of paths defined so far in general do note take the rates by which states are left into account and hence, might exhibit invalid paths. The following measure theoretic concept, however, rules out these paths by assigning probability zero to them. To do so, we need to measure the probability of sets of paths, i.e. of events in an IMC. As a path is the concatenation of a state in S with a sequence of combined transitions $\in \Omega$, we will first define the σ-field of sets of combined transitions, which afterwards can be used to define σ-fields of sets of finite and infinite paths.

Definition 2.3.4 (Events over Paths)
Let $\Omega = Act_{\perp} \times \mathbb{R}_{\geq 0} \times S$ be the set of combined transitions in an IMC $\mathcal{I}$. The σ-field of sets of combined transitions is defined by

$$\mathfrak{F} := \sigma\left(\left\{ (A \times T \times S') \mid A \in 2^{Act_{\perp}}, T \in \mathfrak{B}(\mathbb{R}_{\geq 0}), S' \in 2^S \right\}\right),$$

where $\mathfrak{B}(\mathbb{R}_{\geq 0})$ is the Borel σ-field of $\mathbb{R}_{\geq 0}$, which measures the subsets of $\mathbb{R}_{\geq 0}$.
Then, the σ-field over $Paths^n$ can be defined as follows.

$$\mathfrak{F}_{Paths^n} := \sigma\left(\left\{ S_0 \times M_1 \times ... \times M_n \mid S_0 \in 2^S, M_i \in \mathfrak{F}, 1 \leq i \leq n \right\}\right)$$

A set $\Pi \in \mathfrak{F}_{Paths^n}$ is called a *measurable set of finite paths* of length n. For each such Π we can define its *cylinder*:

$$Cyl(\Pi) := \{ \pi \in Paths^\omega \mid \pi[0..n] \in \Pi \}.$$

A cylinder $Cyl(\Pi)$ is *measurable* if its underlying set of finite paths is measurable, i.e. if $\Pi \in \mathfrak{F}_{Paths^n}$. Then, the σ-field of sets of infinite paths is the minimal σ-field induced by the set of measurable cylinders:

$$\mathfrak{F}_{Paths^\omega} := \sigma\left(\bigcup_{n=0}^{\infty} \{ Cyl(\Pi) \mid \Pi \in \mathfrak{F}_{Paths^n} \}\right).$$

The σ-field $\mathfrak{F}_{Paths^*}$ of sets of finite paths is the smallest σ-field generated by the disjoint union $\biguplus_{n=0}^{\infty} \mathfrak{F}_{Paths^n}$. Finally, the σ-field $\mathfrak{F}_{Paths}$ of sets of finite and infinite paths is the smallest σ-field induced by the disjoint union of the sets of finite and the sets of infinite paths $\biguplus_{n=0}^{\infty} \mathfrak{F}_{Paths^n} \uplus \mathfrak{F}_{Paths^\omega}$.

Definition 2.3.5 (Measurable Rectangles)
A *measurable rectangle* Π of paths of length $n \in \mathbb{N}_{\geq 0}$ is a measurable subset of $Paths^n$ of the form

$$\Pi = S_0 \times A_0 \times T_0 \times ... \times A_{n-1} \times T_{n-1} \times S_n,$$

where $S_i \in 2^S$, $A_i \in 2^{Act_\perp}$, and $T_i \in \mathfrak{B}(\mathbb{R}_{\geq 0})$.

In general, a measurable rectangle is a Cartesian product, where its constituent sets are elements of their respective σ-fields. The set $A \times T \times S \subset \Omega$ of combined transitions is a measurable rectangle, if $A \in 2^{Act_\perp}$, $T \in \mathfrak{B}(\mathbb{R}_{\geq 0})$, and $S \in 2^S$. The set of all measurable rectangles then is denoted by $2^{Act_\perp} \times \mathfrak{B}(\mathbb{R}_{\geq 0}) \times 2^S$ and induces the σ-field $\mathfrak{F}$ of sets of combined transitions (see Definition 2.3.4).
As a next step, we introduce the notion of schedulers, which resolve nondeterministic choices in an IMC. A nondeterministic situation occurs if there are two transitions $s \xrightarrow{\alpha} s_1$ and $s \xrightarrow{\beta} s_2$ and $s_1 \neq s_2$ with $\alpha, \beta \in Act$, i.e. the successor of state s is not uniquely determined. A scheduler then yields a probability distribution over the set $Act(\pi\downarrow) = \{ \alpha, \beta \}$ of enabled actions in s based on the history path $\pi \in Paths^*$ that lead to s.

Definition 2.3.6 (Generic Measurable Scheduler)
A *generic scheduler* on IMC $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$ is a mapping

$$D : Paths^* \times 2^{Act} \mapsto [0, 1],$$

where $D(\pi, \cdot) \in Distr(Act(\pi\downarrow))$ for all $\pi \in Paths^*$ with $\pi\downarrow \in IS$. A generic scheduler is *measurable* (GM scheduler) if and only if for all $A \in 2^{Act}$, $D^{-1}(A) : Paths^* \mapsto [0, 1]$ is measurable.

The measurability condition assures that the scheduler does not resolve the nondeterminism, such that the induced set of paths is non-measurable. Formally, for all $A \in 2^{Act}$ and $B \in \mathfrak{B}([0, 1])$, it holds that $\{ \pi \mid D(\pi, A) \in B \} \in \mathfrak{F}_{Paths^*}$. To obtain a probability distribution over $Act_\perp$, for Markovian states we assume the scheduler to yield probability 1 for action $\perp$, i.e. $D(\pi, \cdot) = \{ \perp \mapsto 1 \}$ if $\pi\downarrow \in MS$. The set of all generic measurable schedulers on $\mathcal{I}$ is denoted by GM.

Definition 2.3.7 (Stationary Scheduler)
A GM scheduler D is called *stationary* if for all $\sigma \in Act_\perp$ and finite paths $\pi, \pi' \in Paths^*$, $\pi\downarrow = \pi'\downarrow$ implies $D(\pi, \sigma) = D(\pi', \sigma)$.

Observe that here the scheduler solely takes the last state of a path into account to compute the probability distribution of the enabled actions. Thus, scheduler D is a function $D : S \times Act_\perp \mapsto [0, 1]$ and we can write $D(s, \alpha)$ instead of $D(\pi, \alpha)$.

Definition 2.3.8 (Deterministic Scheduler)
A GM scheduler D is called deterministic if and only if $D(\pi, \cdot)$ is degenerate for all $\pi \in Paths^*$.

Roughly speaking, D is deterministic if for each finite path $\pi \in Paths^*$ there is an action $\sigma \in Act_\perp$, such that $D(\pi, \sigma) = 1$.

As schedulers resolve nondeterminism, we can now define probability measures for IMCs. We commence with the probability of measurable sets of combined transitions, i.e. subsets of Ω.

Definition 2.3.9 (Probability Measures on Combined Transitions)
Let $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$ be an IMC and $D \in GM$. For all $\pi \in Paths^*$, we define the probability measure $\mu_D(\pi, \cdot) : \mathfrak{F} \mapsto [0, 1]$ by:

$$\mu_D(\pi, M) = \begin{cases} \sum_{\alpha \in Act(s))} \mathbf{1}_M(\alpha, 0, succ(\alpha)) \cdot D(\pi, \{\, \alpha \,\}) & \text{if } s \in IS \\ \int\limits_{\mathbb{R}_{\geq 0}} \mathbf{E}(s) e^{-\mathbf{E}(s)t} \cdot \sum_{s' \in S} \mathbf{1}_M(\bot, t, s') \cdot \mathbf{P}(s, s')\, dt & \text{if } s \in MS, \end{cases}$$

where $s = \pi\!\downarrow$ and $\mathbf{1}_M$ is the indicator with $\mathbf{1}_M(\sigma, t, s') = 1$ if $(\sigma, t, s') \in M$ and 0, otherwise.

Hence, $\mu_D(\pi, M)$ describes the probability obtained considering history π and continuing along one of the combined transitions in $M \in \mathfrak{F}$. If $\pi\!\downarrow$ is an interactive state, $\mu_D(\pi, M)$ yields the probability that scheduler D chooses action $\alpha \in Act(\pi\!\downarrow)$, which leads to the unique successor $succ(\pi\!\downarrow)$, such that the resulting combined transition $(\alpha, 0, succ(\pi\!\downarrow))$ is inherited in the set M. If $\pi\!\downarrow$ is a Markovian transition, the probability to continue according to M is the density for the Markovian transition to execute at time-point t and the probability that a successor state s' of $\pi\!\downarrow$ is reached with $(\bot, t, s') \in M$. The following definition lifts the probability measure $\mu_D(\pi, \cdot)$ on sets of combined transitions to sets of paths of length n, as paths are inductively defined using combined transitions.

Definition 2.3.10 (Probability Measure on Finite Paths)
Let $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$ be an IMC and $D \in GM$. For $n \geq 0$, we define the probability measures Pr_D^n inductively on the measurable space $(Paths^n, \mathfrak{F}_{Paths^n})$:

$$\begin{aligned} Pr_D^0 : \mathfrak{F}_{Paths^0} \mapsto [0,1] : \quad & Pr_D^0(\Pi) = 1 \text{ if } s_0 \in \Pi \\ & Pr_D^0(\Pi) = 0 \text{ otherwise} \\ Pr_D^n : \mathfrak{F}_{Paths^n} \mapsto [0,1] : \quad & \Pi \mapsto \int_{Paths^{n-1}} Pr_D^{n-1}(d\pi) \int_{\Omega} \mathbf{1}_{\Pi}(\pi \circ m)\, \mu_D(\pi, dm), \end{aligned}$$

where $\mathbf{1}_\Pi$ is an indicator function with $\mathbf{1}_\Pi(\pi \circ m) = 1$ if $\pi \circ m \in \Pi$ and 0 otherwise, and $\circ$ is the concatenation of path π with the combined transition m.

Given a set of paths $\Pi \in \mathfrak{F}_{Paths^n}$ of length $n \in \mathbb{N}_{\geq 0}$, Pr_D^n computes the probability that one of the paths in Π is chosen by multiplying the probabilities $Pr_D^{n-1}(d\pi)$ of path prefixes π of length $(n-1)$ with the probability $\mu_D(\pi, dm)$ of a combined transition $m \in \Omega$ which extends π to a path in Π. The functions Pr_D^n thus yield measures on all σ-fields $\mathfrak{F}_{Paths^n}$ of subsets of paths of length n, which extends to a measure on $(Paths^\omega, \mathfrak{F}_{Paths^\omega})$: if $B \in \mathfrak{F}_{Paths^n}$ is a measurable base and $C = Cyl(B)$, we define $Pr_D^\omega(C) = Pr_D^n(B)$. Due to the inductive definition of Pr_D^n, the Ionescu-Tulcea extension theorem applies (see [18, 23]), which yields a unique extension of Pr_D^ω to arbitrary sets in $\mathfrak{F}_{Paths^\omega}$. Note that $Pr_{s,D}^n(\Pi)$ and $Pr_{s,D}^\omega(\Pi)$ denote the probabilities that finite, resp., infinite path $\pi \in \Pi$ is chosen, where all paths in Π start in state s.

3. Probabilistic Backward Bisimulation

In this chapter we will present the formal definition of the strong binary relation *probabilistic backward bisimulation, pbb* for short, which relates pairs of states of a single IMC or of pairs of IMCs. We will use the notion of probabilistic backward bisimulation to define a quotient system, i.e. an abstraction that minimizes the state space of the original IMC and investigate in the following chapter whether it preserves some timed properties, such that analysis tools can be applied on the diminished state space more efficiently. Furthermore, we will check if the reduced model is a correct abstraction of the original one in the sense of probabilistic backward bisimulation. However, assuming that minimization is based on a partition-refinement technique, this is not an on-the-fly algorithm but instead requires the entire state space of the model that is to be analyzed in advance [13]. This is a tremendous disadvantage, since realistic systems consist of millions of states. In case of the forward variant of probabilistic bisimulation, this can be compensated thanks to the compositional nature of IMCs and the fact that the forward version is a congruence with respect to parallel composition, as this allows for a compositional generation and reduction of the original system, resulting in a smaller system that is bisimilar to the original one. Unfortunately, this does not hold for pbb, since original and quotient system are in general not bisimilar. Last but not least, we will give an overview of the comparison of probabilistic forward [13] and backward bisimulation on IMCs.

3.1. PBB Equivalence

To compare two or more states of an IMC, we now introduce the notion of probabilistic backward bisimulation, which is based on the forward approach for IMCs in [13] and the backward variant for CTMCs in [21]. The intuition is that two states s and t are bisimilar if their predecessors can mutually mimic their interactive behavior with respect to the reachability of s and t, i.e. for each transition $s' \xrightarrow{\alpha} s$ there is an interactive transition $t' \xrightarrow{\alpha} t$, such that s' and t' are bisimilar. Recall that internal interactive transitions take precedence over Markovian transitions (see Definition 2.2.3). This is due to the fact that internal transitions do not interact with the environment and hence will not be delayed. As a consequence, they are always executed immediately, whereas the probability of a Markovian transition to be executed without any delay is zero. According to that and since an IMC will be closed for analyzation purposes, either both s and t have outgoing interactive transitions or none of them has. Furthermore, the cumulative rate by which some equivalence class C of states can reach states s and t within one step must be equal, i.e. $\mathbf{R}(C, s) = \sum_{u \in C} \mathbf{R}(u, s) = \sum_{u \in C} \mathbf{R}(u, t) = \mathbf{R}(C, t)$. However, there is no need to require equality of the cumulative rates by which s and t are reached from C, if C has an outgoing internal interactive transition. The same holds for the exit rates of s and t, which must be equal in case of absence of outgoing internal interactive transitions.

Definition 3.1.1 (Probabilistic Backward Bisimulation)
Let $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$ be an IMC. An equivalence relation $\mathcal{R} \subseteq S \times S$ is a *probabilistic backward bisimulation*, *pbb* for short, on $\mathcal{I}$ if for any $(s,t) \in \mathcal{R}$ and equivalence classes $C \in S/\mathcal{R}$ the following holds:

(1) $L(s) = L(t)$,

(2) for any $\alpha \in Act$, $\mathbf{T}(C, \alpha, s) = \mathbf{T}(C, \alpha, t)$,

(3) $Post_\alpha(s) = \emptyset$ if and only if $Post_\alpha(t) = \emptyset$,

(4) if $Post_\tau(C) = \emptyset$ then $\mathbf{R}(C, s) = \mathbf{R}(C, t)$, and

(5) if $Post_\tau(s) = Post_\tau(t) = \emptyset$ then $\mathbf{E}(s) = \mathbf{E}(t)$,

where $\mathbf{T}(C, \alpha, s) = 1$ if and only if $\{ s' \in C \mid s' \xrightarrow{\alpha} s \}$ is non-empty and $\mathbf{R}(C, s) = \sum_{s' \in C} \mathbf{R}(s', s)$ is the cumulative rate to go from C to s.

States s and t are *probabilistic backward bisimulation-equivalent or bisimilar (pbb, for short)*, denoted $s \sim^b t$, if there exists a probabilistic backward bisimulation $\mathcal{R}$ for $\mathcal{I}$ with $(s,t) \in \mathcal{R}$.

Recall that by definition the initial state s_0 is labeled by the special atomic proposition $z \notin AP$ (see page 4) and consequently, the equivalence class containing s_0 will always be a singleton. This is due to the fact that s_0 is the only state that can have an empty past and pbb equivalent states are expected to mimic each others (interactive and Markovian) behavior in a backward manner. Condition (1) is the *labeling* condition, which states that equivalent states must have the same labeling. The second requirement assures that if s is reachable via some interactive transition labeled with action $\alpha \in Act$ from equivalence class $C \in S/\mathcal{R}$ then so is t, and vice versa. The third statement is especially interesting in context with the analysis of path properties of closed IMCs or IMCs to which the concept of hiding (see Definition 2.2.7) is applied. As an example, consider Figure 3.1, where state s_5 is the only b-labeled state. Following conditions (1), (2), (4), and (5), states s_1 and s_4 would lie in the same equivalence class. Assume that the quotient system, i.e. the system that groups equivalent states into a single one, is to be computed. The resulting system can be observed in Figure 3.2, where we obtain a single state s_1, s_4 for the equivalent states s_1 and s_4. Furthermore, suppose that the system will be closed (see Figure 3.3) as some property analyses are to be performed. In that case, the outgoing α-transition of state s_1, s_4 will be *internal*. As a consequence of the maximal progress assumption (see Definition 2.2.3), the Markovian transition of s_4, which leads to the b-labeled state s_5, as well as the Markovian transition of state s_1, will be eliminated as they will never be executed due to the outgoing *internal* α-transition of state s_1, s_4. Furthermore, since the Markovian transition of state s_4 is the only incoming transition of s_5, it would no longer be reachable, although when closing the *original* system, state s_5 would still be reachable from state s_4. Thus, to prevent this crucial loss of information, we need to introduce *at least* the semi-forward bisimulation condition

(3) of Definition 3.1.1. It is semi-forward since we are not interested in the target states and the action-label of the transition, but only in its existence. Note that since every IMC will be closed if property-analyses are to be performed, it does not suffice to require that equivalent states either all or none have outgoing internal interactive transitions, but we must range over *all* interactive transitions.

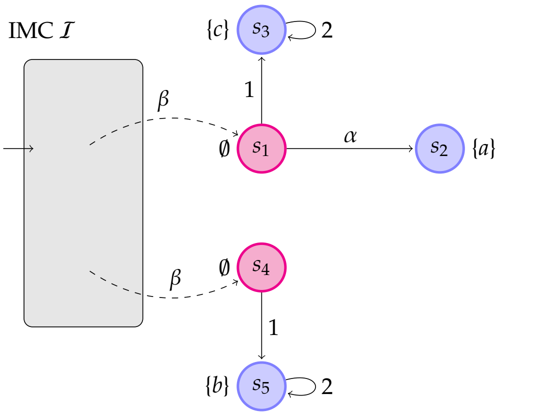

Figure 3.1.: Original IMC $\mathcal{I}$ with Equivalent States s_1 and s_4

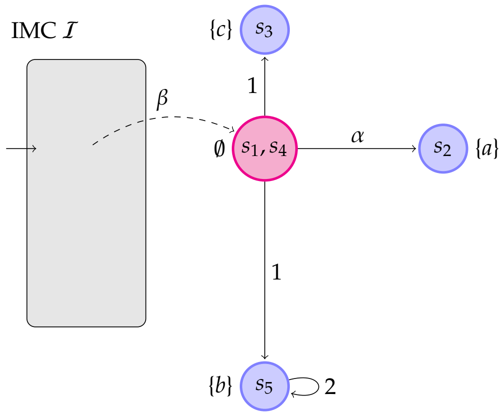

Figure 3.2.: Quotient After Merging Equivalent States

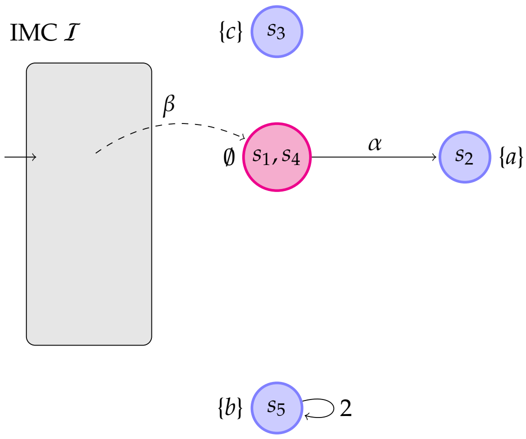

Figure 3.3.: Closing the Quotient IMC

Condition (4) requires that whenever the cumulative rate to reach state s from some equivalence class $C \in S/\mathcal{R}$, from which no internal interactive transition emanates, is $\mathbf{R}(C, s) = \lambda$, then the cumulative rate to reach t from C is λ, and vice versa. The last requirement states that if s and t have no outgoing internal interactive transitions, then they must have the same exit rate E. Similar to condition (3), this imposes an analogy on the one-step forward or future behavior of equivalent states. The basic idea is to assure that in the quotient system of the original IMC, the rate of moving from equivalence class C_1 to a class

C_2 is the *average value* of moving from some state in C_1 to some state in C_2 in the *original* IMC.

The notion of pbb equivalent states of a single IMC can be adapted to one of pairs of IMCs $(\mathcal{I}_1, \mathcal{I}_2)$, which allows to compare two IMCs with respect to the probabilistic backward bisimulation relation. For that, we additionally need to require that the initial states of the IMCs are reachable in a pbb-equivalent manner.

Definition 3.1.2 (Probabilistic Backward Bisimilar IMCs)
Let $\mathcal{I}_i = (S_i, Act, \longrightarrow_i, \Longrightarrow_i, s_{0,i}, AP, L_i)$, $i = 1, 2$, be two IMCs. A *probabilistic backward bisimulation* for $(\mathcal{I}_1, \mathcal{I}_2)$ is an equivalence relation $\mathcal{R} \subseteq S_1 \times S_2$, such that

(A) $(s_{0,1}, s_{0,2}) \in \mathcal{R}$,

(B) for all $(s_1, s_2) \in \mathcal{R}$ and equivalence classes $C \in (S_1 \uplus S_2)/\mathcal{R}$

  (1) $L(s_1) = L(s_2)$,

  (2) for any $\alpha \in Act$, $\mathbf{T}(C, \alpha, s_1) = \mathbf{T}(C, \alpha, s_2)$,

  (3) $Post_\alpha(s_1) = \emptyset$ if and only if $Post_\alpha(s_2) = \emptyset$,

  (4) if $Post_\tau(C) = \emptyset$ then $\mathbf{R}(C, s_1) = \mathbf{R}(C, s_2)$, and

  (5) if $Post_\tau(s_1) = Post_\tau(s_2) = \emptyset$ then $\mathbf{E}(s_1) = \mathbf{E}(s_2)$,

where $(S_1 \uplus S_2)/\mathcal{R}$ denotes the state space $S_1 \uplus S_2$ with respect to $\mathcal{R}$, i.e. the set of all pbb-equivalence classes under $\mathcal{R}$.

$\mathcal{I}_1$ and $\mathcal{I}_2$ are *pbb*, denoted $\mathcal{I}_1 \sim^b \mathcal{I}_2$, if there exists a probabilistic backward bisimulation $\mathcal{R}$ for $(\mathcal{I}_1, \mathcal{I}_2)$.

Since a path of an IMC starts in its initial state, it is natural to require that the initial states of two IMCs, that are candidates to be pbb-equivalent, must be able to mimic each others backward behavior and cannot be bisimilar to any non-initial state. Example 3.1.1 illustrates the above definition.

Example 3.1.1 (PBB IMCs)
Consider the open IMCs $\mathcal{I}_1$ (left) and $\mathcal{I}_2$ (right) given in Figure 3.4 with initial states s_0 and t_0, respectively, and internal action $Act_i = \{ \tau \}$. For the sake of simplicity, assume that $L(s) = \emptyset$ for all $s \in ((S_1 \backslash \{ s_{0,1} \}) \cup (S_2 \backslash \{ s_{0,2} \}))$, $L(s_{0,1}) = L(s_{0,2}) = \{ z \}$ (by definition, see page 4). There exists a probabilistic backward bisimulation for $(\mathcal{I}_1, \mathcal{I}_2)$:

$$\mathcal{R} = \{ (s_0, t_0), (s_1, t_1), (s_2, t_2), (s_3, t_2), (s_4, t_3), (s_5, t_3) \}.$$

Furthermore, we have four equivalence classes $C_0 = \{ s_0, t_0 \}$, $C_1 = \{ s_1, t_1 \}$, $C_2 = \{ s_2, s_3, t_2 \}$, and $C_3 = \{ s_4, s_5, t_3 \}$. We will now show that $\mathcal{R}$ is a pbb by checking the conditions required in Definition 3.1.2 for each pair in $\mathcal{R}$. Note that since all non-initial states and both initial states have the same labeling, condition (B.1) is trivially satisfied. States s_0 and t_0 have no incoming interactive transitions and hence, condition (B.2) is fulfilled. Since both have at least one outgoing (internal) interactive transition, condition (B.3) holds. Either state is solely reachable via a Markovian self-loop, but as these are the only incoming transitions and since $Post_\tau(s_0) \neq \emptyset \neq Post_\tau(t_0)$, the remaining conditions hold immediately. States s_1, t_1 have an incoming τ-transition from s_0 and t_0, respectively. Since s_0 and t_0 are in the same equivalence class C_0 and since these are the only incoming interactive transitions, condition (B.2) is satisfied. Furthermore, both states have at least one outgoing interactive transition, such that condition (B.3) is fulfilled. Both states have no outgoing internal transitions but as they are neither reachable by nor have an emanating Markovian transition, conditions (B.4) and (B.5) trivially hold. For pair (s_2, t_2), we have that $\mathbf{T}(C_0, \alpha, s_2) = \mathbf{T}(C_0, \alpha, t_2)$ and $\mathbf{T}(C_1, \alpha, s_2) = \mathbf{T}(C_1, \alpha, t_2)$. Since $Post_\alpha(s_2) = Post_\alpha(t_2) = \emptyset$, condition (B.3) is satisfied. Since $Post_\tau(C_0) \neq \emptyset \neq Post_\tau(C_3)$ and since neither of the states is reachable by a Markovian transition from class C_1, (B.4) holds. Neither s_2 nor t_2 has an outgoing internal interactive transition and since $\mathbf{E}(s_2) = 8 = \mathbf{E}(t_2)$, it follows that s_2 and t_2 are pbb. A similar argumentation holds for the pair (s_3, t_2). Considering the pair (s_4, t_3), requirement (B.2) is satisfied, since $\mathbf{T}(C_3, \tau, s_4) =$

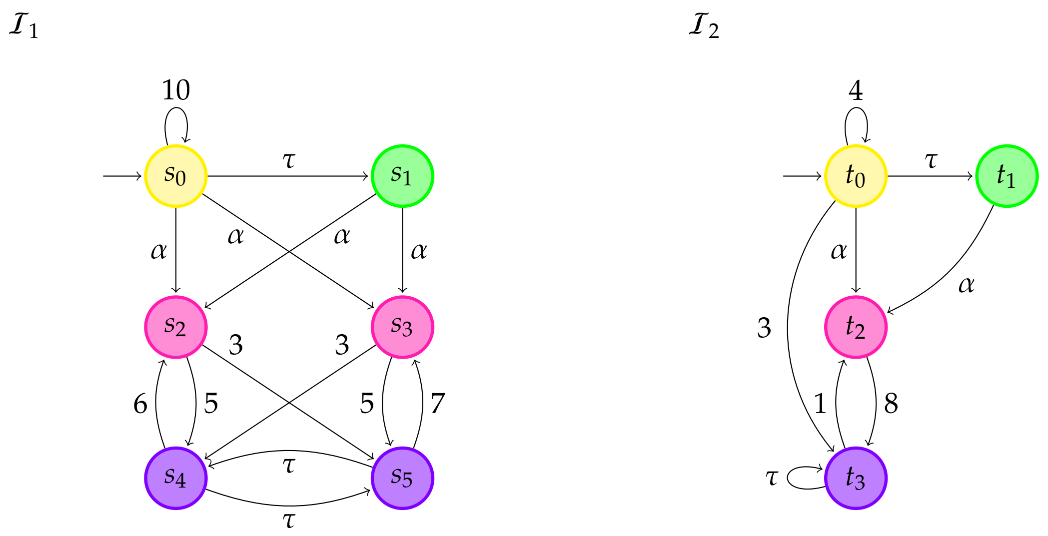

Figure 3.4.: Probabilistic Backward Bisimilar IMCs

$\mathbf{T}(C_3, \tau, t_3)$. Both states have an outgoing τ-transition and consequently, (B.3) and (B.5) hold. Condition (B.4) is true, as $\mathbf{R}(C_2, s_4) = \mathbf{R}(C_2, t_3) = 8$ and $Post_\tau(C_3) \neq \emptyset$. For states s_5, t_3, the requirements can be verified in the same way and since $(s_0, t_0) \in \mathcal{R}$, the initial condition (A) holds and it follows that $\mathcal{R}$ is a pbb for $(\mathcal{I}_1, \mathcal{I}_2)$.
Note that if τ was not internal, then none of the states are pbb. For example, consider states s_0 and t_0. Since $\mathbf{E}(s_0) = 10 \neq 7 = \mathbf{E}(t_0)$ and $\mathbf{R}(\{s_0, t_0\}, s_0) = 10 \neq 4 = \mathbf{R}(\{s_0, t_0\}, t_0)$, they cannot be bisimilar. Furthermore, states s_2 and s_3 do not lie in the same equivalence class, since s_4 and s_5 are not equivalent and s_4 reaches s_2 with rate 6 but not s_3, and s_5 reaches s_3 with rate 7 but not s_2. □

Lemma 3.1.1 (Coarsest PBB)
For IMC $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$ it holds that

$$\sim^b \text{ is the coarsest probabilistic backward bisimulation for } \mathcal{I}.$$

Proof Let $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$ be an IMC and s_1, s_2 be two states in S. Let $\mathcal{R}$ be a pbb on $\mathcal{I}$ and assume that the pair (s_1, s_2) is included in $\mathcal{R}$, but $s_1 \nsim^b s_2$. Then, one of the conditions in Definition 3.1.1 must be violated. This contradicts the assumption that $\mathcal{R}$ is a pbb containing the pair of states (s_1, s_2). It immediately follows that each pbb $\mathcal{R}$ for $\mathcal{I}$ is finer than $\sim^b$ and that $\sim^b$ is the coarsest pbb. ∎

We have already mentioned the concept of a component-wise minimization of a given IMC $\mathcal{I}$. This model is the parallel composition (see Definition 2.2.6) of finitely many IMCs that synchronize via synchronization actions in A_i, i.e.

$$\mathcal{I} = \mathcal{I}_1 \parallel_{A_1} \mathcal{I}_2 \parallel_{A_2} \dots \parallel_{A_{n-1}} \mathcal{I}_n,\ n \in \mathbb{N}.$$

Since each IMC $\mathcal{I}_i$ is subject to synchronization, they will be open IMCs. Theorem 3.1.1 shows that probabilisitic backward bisimulation is substitutive with respect to parallel composition, i.e. if two open IMCs $\mathcal{I}_1, \mathcal{I}_2$ over the same set of actions are pbb then the respective parallel compositions with some IMC $\mathcal{I}_3$ are pbb.

Theorem 3.1.1 ($\sim^b$ is a congruence w.r.t. parallel composition)
Let $\mathcal{I}_i = (S_i, Act_i, \longrightarrow_i, \Longrightarrow_i, s_{0,i}, AP, L_i)$, $i \in \{1, 2, 3\}$, be three open IMCs with $Act_1 = Act_2$ and a set $A \subseteq Act_1 \cap Act_3$ of synchronization actions. Then,

$$\mathcal{I}_1 \sim^b \mathcal{I}_2 \text{ implies } \mathcal{I}_1 \parallel_A \mathcal{I}_3 \sim^b \mathcal{I}_2 \parallel_A \mathcal{I}_3.$$

Proof Let $\mathcal{I}_i = (S_i, Act_i, \longrightarrow_i, \Longrightarrow_i, s_{0,i}, AP, L_i)$, $i \in \{1, 2, 3\}$, be three IMCs, A a set of synchronization actions as before and $\mathcal{I}_1 \sim^b \mathcal{I}_2$. Furthermore, let $\mathcal{I}_{1,3} = \mathcal{I}_1 \parallel_A \mathcal{I}_3$ and $\mathcal{I}_{2,3} = \mathcal{I}_2 \parallel_A \mathcal{I}_3$ and let $\mathcal{R} = \{ ((s_1, s_3), (s_2, s_3)) \mid s_1 \in S_1, s_2 \in S_2, s_3 \in S_3, s_1 \sim^b s_2 \}$. We have to show that $\mathcal{R}$ is a pbb for $(\mathcal{I}_{1,3}, \mathcal{I}_{2,3})$, i.e. each pair of states in $\mathcal{R}$ satisfies the conditions of Definition 3.1.2.
Let $((s_1, s_3), (s_2, s_3))$ be a pair in $\mathcal{R}$.

(A) Since $s_{0,1} \sim^b s_{0,2}$, it holds by definition that the pair of states $((s_{0,1}, s_{0,3}), (s_{0,2}, s_{0,3}))$ is included in $\mathcal{R}$.

(B.1) By definition of the parallel composition, condition (B.1) is trivially satisfied.

(B.2) Statement (B.2) requires that whenever there is an equivalence class $C \in ((S_1 \times S_3) \uplus (S_2 \times S_3))/\mathcal{R}$ that can reach state (s_1, s_3) in one step via an interactive transition labeled by some action $\alpha \in Act_1 \cup Act_3$, then state (s_2, s_3) is also reachable from C in one step via an α-transition, and vice versa. We have to distinguish three cases:

Case 1: W.l.o.g. assume there exists $(s_1', s_3') \in (S_1 \times S_3)$, such that $(s_1', s_3') \xrightarrow{\alpha} (s_1, s_3)$ is an interactive transition in $\mathcal{I}_{1,3}$ and $\alpha \in A$. Since $s_1 \sim^b s_2$, there exists a state $s_2' \in S_2$, such that $s_1' \sim^b s_2'$ and $s_2' \xrightarrow{\alpha} s_2$ is an interactive transition in $\mathcal{I}_2$. Hence, by definition of the interactive transition relation of the parallel composition, there exists a transition $(s_2', s_3') \xrightarrow{\alpha} (s_2, s_3)$ in $\mathcal{I}_{2,3}$. Furthermore, it holds that the pair of states $((s_1', s_3'), (s_2', s_3'))$ is in $\mathcal{R}$, i.e. they belong to the same equivalence class in $((S_1 \times S_3) \uplus (S_2 \times S_3))/\mathcal{R}$.

Case 2: W.l.o.g. assume there exists $(s_1', s_3) \in (S_1 \times S_3)$, such that $(s_1', s_3) \xrightarrow{\alpha} (s_1, s_3)$ is an interactive transition in $\mathcal{I}_{1,3}$ and $\alpha \notin A$. Since $s_1 \sim^b s_2$, it immediately follows that there exists a state $s_2' \in S_2$, such that $s_1' \sim^b s_2'$ and $(s_2', s_3) \xrightarrow{\alpha} (s_2, s_3)$ is an interactive transition in $\mathcal{I}_{2,3}$. Furthermore, by definition it holds that $((s_1', s_3), (s_2', s_3)) \in \mathcal{R}$.

Case 3: W.l.o.g. assume there exists $(s_1, s'_3) \in (S_1 \times S_3)$, such that $(s_1, s'_3) \xrightarrow{\alpha} (s_1, s_3)$ is an interactive transition in $\mathcal{I}_{1,3}$ and $\alpha \notin A$. It directly follows that there is an interactive transition $(s_2, s'_3) \xrightarrow{\alpha} (s_2, s_3)$ in $\mathcal{I}_{2,3}$ and we have that $((s_1, s'_3), (s_2, s'_3)) \in \mathcal{R}$.

Consequently, condition (B.2) is satisfied by all pairs of states in $\mathcal{R}$.

(B.3) Assume w.l.o.g. that there is an interactive transition $(s_1, s_3) \xrightarrow{\alpha} (s'_1, s'_3)$ in IMC $\mathcal{I}_{1,3}$. Then, if $\alpha \in A$, we have that $Post_\alpha(s_1) = Post_\alpha(s_2) \neq \emptyset$, since $s_1 \sim^b s_2$. It follows that there exists a state $s'_2 \in S_2$, such that $(s_2, s_3) \xrightarrow{\alpha} (s'_2, s'_3)$ is an interactive transition in IMC $\mathcal{I}_{2,3}$. If $\alpha \notin A$, there are two cases. First, either $s_3 = s'_3$ and the same argumentation as in the case when α is a synchronization action applies. Second, $s_1 = s'_1$ and it directly follows that there is a transition $(s_2, s_3) \xrightarrow{\alpha} (s_2, s'_3)$ in IMC $\mathcal{I}_{2,3}$.

(B.4) Let C be an equivalence class in $((S_1 \times S_3) \uplus (S_2 \times S_3))/\mathcal{R}$, i.e. a set of states $(s_1, s_3), (s_2, s_3)$, such that for all $s_1 \in S_1, s_2 \in S_2$, it holds that $s_1 \sim^b s_2$. Assume that $Post_\tau(C) = \emptyset$ and let $((s_1, s_3), (s_2, s_3)) \in \mathcal{R}$. We have to prove that $\mathbf{R}(C, (s_1, s_3)) = \mathbf{R}(C, (s_2, s_3))$.

$$
\begin{aligned}
& \mathbf{R}(C, (s_1, s_3)) \\
& \text{(* by definition of } \mathbf{R} \text{ *)} \\
= \quad & \sum\nolimits_{(t,t') \in C} \mathbf{R}((t, t'), (s_1, s_3)) \\
& \text{(* by definition of the Markovian transition relation of the} \\
& \text{ parallel composition *)} \\
= \quad & \sum\nolimits_{(t,t') \in C} \mathbf{R}(t, s_1) + \sum\nolimits_{(t,t') \in C} \mathbf{R}(t', s_3) \\
& \text{(* since } s_1 \sim^b s_2 \text{ *)} \\
= \quad & \sum\nolimits_{(t,t') \in C} \mathbf{R}(t, s_2) + \sum\nolimits_{(t,t') \in C} \mathbf{R}(t', s_3) \\
= \quad & \sum\nolimits_{(t,t') \in C} \mathbf{R}((t, t'), (s_2, s_3)) \\
& \text{(* by definition of } \mathbf{R} \text{ *)} \\
= \quad & \mathbf{R}(C, (s_2, s_3))
\end{aligned}
$$

(B.5) Let $((s_1, s_3), (s_2, s_3)) \in \mathcal{R}$ and assume (w.l.o.g.) that $Post_\tau((s_1, s_3)) = \emptyset$. By definition of the Markovian transition relation of the parallel composition, we have that $\mathbf{E}((s_1, s_3)) = \mathbf{E}(s_1) + \mathbf{E}(s_3)$. Since $s_1 \sim^b s_2$, it follows that $\mathbf{E}((s_1, s_3)) = \mathbf{E}(s_1) + \mathbf{E}(s_3) = \mathbf{E}(s_2) + \mathbf{E}(s_3) = \mathbf{E}((s_2, s_3))$.

It follows that $\mathcal{R}$ is an pbb for $(\mathcal{I}_{1,3}, \mathcal{I}_{2,3})$. ∎

The next Example 3.1.2 computes the parallel composition of two IMCs $\mathcal{I}_1$ and $\mathcal{I}_2$ with (resp.) a third IMC $\mathcal{I}_3$ and gives the resulting probabilistic backward bisimulation relation for the two composed systems.

Example 3.1.2 (Congruence w.r.t. Parallel Composition)
Consider Figure 3.5, where τ is the only internal interactive transition. The open IMCs $\mathcal{I}_1$ and $\mathcal{I}_2$ are probabilistic backward bisimilar and the binary relation

$$\mathcal{R} = \{ (s_0, t_0), (s_1, t_1), (s_1, t_3), (s_3, t_1), (s_3, t_3), (s_2, t_2) \}$$

is a pbb for the pair $(\mathcal{I}_1, \mathcal{I}_2)$. Then, each of them will be composed in parallel with IMC $\mathcal{I}_3$ over the set $A = \{ \alpha, \beta \}$ of synchronization actions, i.e. $\mathcal{I}_{1,3} = \mathcal{I}_1 \,\|_A\, \mathcal{I}_3$ and $\mathcal{I}_{2,3} = \mathcal{I}_2 \,\|_A\, \mathcal{I}_3$ are to be computed.

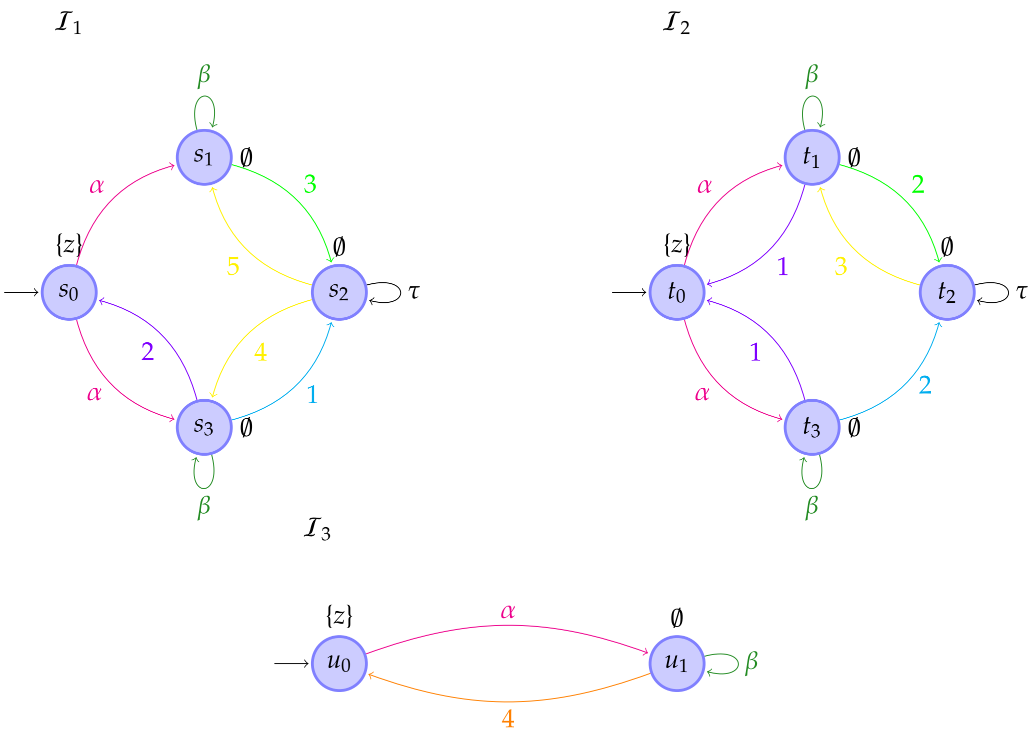

Figure 3.5.: Two pbb IMCs $\mathcal{I}_1$ and $\mathcal{I}_2$ will be composed in parallel with IMC $\mathcal{I}_3$

Following Theorem 3.1.1, the binary relation

$$\begin{aligned}
\mathcal{R}_\| = \{ \; & ((s_0, u_0), (t_0, u_0)), \\
& ((s_0, u_1), (t_0, u_1)), \\
& ((s_1, u_0), (t_1, u_0)), ((s_1, u_0), (t_3, u_0)), ((s_3, u_0), (t_1, u_0)), ((s_3, u_0), (t_3, u_0)), \\
& ((s_1, u_1), (t_1, u_1)), ((s_1, u_1), (t_3, u_1)), ((s_3, u_1), (t_1, u_1)), ((s_3, u_1), (t_3, u_1)), \\
& ((s_2, u_0), (t_2, u_0)), \\
& ((s_2, u_1), (t_2, u_1)) \; \}
\end{aligned}$$

is a pbb for the pair $(\mathcal{I}_{1,3}, \mathcal{I}_{2,3})$.
Figure 3.6 shows the resulting IMCs after parallel composition, where states that have similar coloring belong to the same equivalence classes in $(S_{1,3} \uplus S_{2,3})/\mathcal{R}_\|$.

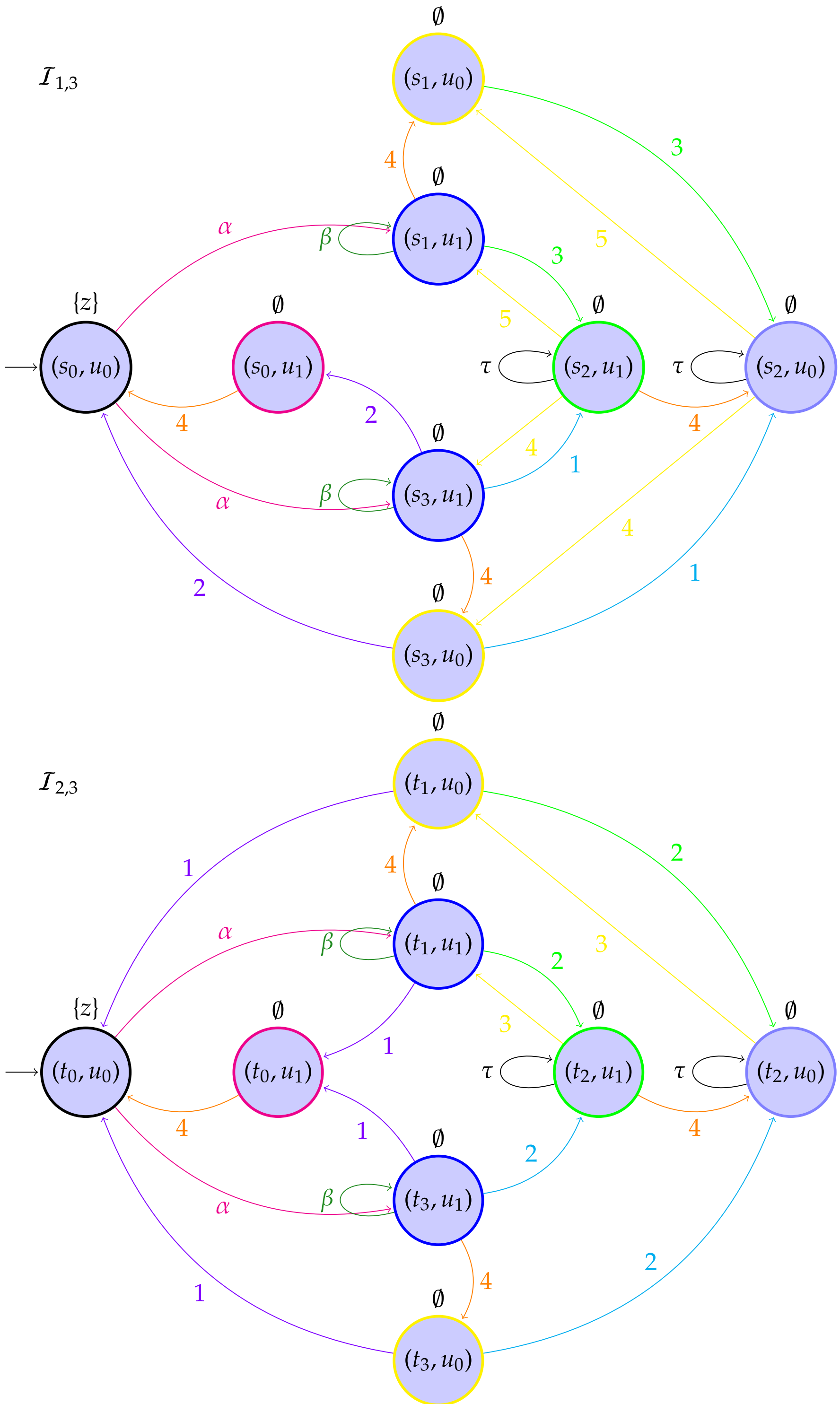

Figure 3.6.: Parallel composition is pbb

3.2. Quotient System under $\sim^b$

Probabilistic backward bisimulation defines an equivalence relation on the states of a given IMC. Equivalent states reveal an equivalent backward behavior in the sense of pbb and thus, they can be merged into a single one to eliminate redundancies and to reduce the state space of the given IMC (hopefully) significantly. Unfortunately, the thus computed compressed system is *not* an *exact* abstraction of the original one, as the computation of the rates reveals average values and thus, condition (B.4) in Definition 3.1.2 is not preserved exactly. However, the quotient's state space might be significantly smaller than the original one's and the idea is (see the following chapter) to analyze the compressed system on properties of interest with a smaller complexity concerning time and space.

Definition 3.2.1 (Quotient System under $\sim^b$)
For IMC $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$ the probabilistic backward bisimulation quotient IMC $\mathcal{I}/\!\sim^b$ is defined as follows.

$$\mathcal{I}/\!\sim^b = (S/\!\sim^b, Act, \longrightarrow_{\sim^b}, \Longrightarrow_{\sim^b}, [s_0]_{\sim^b}, AP, L_{\sim^b}),$$

where

- $\longrightarrow_{\sim^b}$ is defined by $\dfrac{s \xrightarrow{\alpha} t}{[s]_{\sim^b} \xrightarrow{\alpha}_{\sim^b} [t]_{\sim^b}}$, $\alpha \in Act$,
- $\Longrightarrow_{\sim^b}$ is defined by

$$\frac{s \overset{\lambda_1}{\Longrightarrow} t}{[s]_{\sim^b} \overset{\lambda_2}{\Longrightarrow}_{\sim^b} [t]_{\sim^b}}, \text{ with } \lambda_2 = \frac{\sum_{s\in[s]_{\sim^b}} \sum_{t\in[t]_{\sim^b}} \mathbf{R}(s,t)}{\mid [s]_{\sim^b} \mid}, \lambda_1, \lambda_2 \in \mathbb{R}_{>0},$$

  and
- $L_{\sim^b}([s]_{\sim^b}) = L(s)$.

Each state in the quotient system consists of all states that belong to the same equivalence class under $\sim^b$. The initial state $[s_0]_{\sim^b}$ of $\mathcal{I}/\!\sim^b$ is the equivalence class containing the initial state s_0 of the original IMC $\mathcal{I}$. Since the definition of $\sim^b$ requires that equivalent states are reachable via the same interactive transitions, equivalence class $[t]_{\sim^b}$ in the quotient system is reached via $\alpha \in Act$ from class $[s]_{\sim^b}$ if there exists a transition leading from s to t with label α in the original IMC. If there is a Markovian transition from s to t in the original model, then there exists a Markovian transition in the quotient model, which leads from class $[s]_{\sim^b}$ to class $[t]_{\sim^b}$. The rate of this transition is the average value of all Markovian transitions leading from all states of the class $[s]_{\sim^b}$ to all states of class $[t]_{\sim^b}$ in the original system. Hence, no restrictions are imposed on the rate of a transition emanating from a *single* state s to t, but rather on the *cumulative* rate of all transitions from all states in the class $[s]_{\sim^b}$ to state t. Last but not least, the label of the class equals the label of the states included in that class.
The following Theorem 3.2.1 shows that Definition 3.2.1 of the abstract system is **not** an exact reduction in the sense that the resulting quotient is probabilistic backward bisimilar to the original one.

Theorem 3.2.1 (Non-PBB Equivalence of $\mathcal{I}$ and $\mathcal{I}/\sim^b$)
Let $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$, be an IMC and $\mathcal{I}/\sim^b = (S/\sim^b, Act, \longrightarrow_{\sim^b}, \Longrightarrow_{\sim^b}, [s_0]_{\sim^b}, AP, L_{\sim^b})$ its quotient system under $\sim^b$. Then

$$\mathcal{I} \not\sim^b \mathcal{I}/\sim^b$$

in general.

Proof Let $\mathcal{I}$ be an IMC and $\mathcal{I}/\sim^b$ its pbb quotient system. We will first show that conditions (A) through (B.3) and (B.5) of Definition 3.1.2 are satisfied for the pair $(\mathcal{I}, \mathcal{I}/\sim^b)$. Therefore, let $\mathcal{R} = \{\ (s, [s]_{\sim^b}) \mid s \in S)\ \}$. An equivalence class $C \in (S \uplus S/\sim^b)/\mathcal{R}$ is defined as $C = \{\ s_1, ..., s_n, [s]_{\sim^b} \mid [s]_{\sim^b} \in S/\sim^b, s_i \in [s]_{\sim^b}, n = |[s]_{\sim^b}|\ \}$. By definition of the quotient system under $\sim^b$, it can easily be seen that conditions (A) through (B.3) of Definition 3.1.2 are satisfied by all pairs of states in $\mathcal{R}$. Requirement (B.5) states that if $\sim^b$-equivalent states have no outgoing internal interactive transitions then they must have the same exit rate $\mathbf{E}$. Let C_s be an equivalence class under $\mathcal{R}$ containing state $s \in S$ and assume that $Post_\tau(s) = \emptyset$. Then, $Post_\tau(s') = \emptyset$ and $\mathbf{E}(s) = \mathbf{E}(s')$ for all states $s' \in C_s \backslash \{\ [s]_{\sim^b}\ \}$. Furthermore, the exit rate of state $[s]_{\sim^b}$ is equal to the exit rate of state s, and thus, to all states $s' \in [s]_{\sim^b}$, which can be seen as follows.

$$\begin{aligned}
& \mathbf{E}([s]_{\sim^b}) \\
& \text{(* by Definition of the exit rate } \mathbf{E} \text{ *)} \\
= \; & \sum\nolimits_{[t]_{\sim^b} \in S/\sim^b} \mathbf{R}([s]_{\sim^b}, [t]_{\sim^b}) \\
& \text{(* according to Definition (3.2.1) *)} \\
= \; & \sum\nolimits_{[t]_{\sim^b} \in S/\sim^b} \left(\frac{1}{\mid [s]_{\sim^b} \mid} \left(\sum\nolimits_{s \in [s]_{\sim^b}} \sum\nolimits_{t \in [t]_{\sim^b}} \mathbf{R}(s,t) \right) \right) \\
& \text{(* by commutativity *)} \\
= \; & \frac{1}{\mid [s]_{\sim^b} \mid} \left(\sum\nolimits_{s \in [s]_{\sim^b}} \sum\nolimits_{[t]_{\sim^b} \in S/\sim^b} \sum\nolimits_{t \in [t]_{\sim^b}} \mathbf{R}(s,t) \right) \\
& \text{(* since } S/\sim^b \text{ is a partition of } S \text{ *)} \\
= \; & \frac{1}{\mid [s]_{\sim^b} \mid} \left(\sum\nolimits_{s \in [s]_{\sim^b}} \sum\nolimits_{t \in S} \mathbf{R}(s,t) \right) \\
& \text{(* since } \mathbf{E}(s_i) = \mathbf{E}(s_j) \text{ for all } s_i, s_j \in [s]_{\sim^b} \text{ *)} \\
= \; & \sum\nolimits_{t \in S} \mathbf{R}(s,t) \\
& \text{(* by Definition of the exit rate } \mathbf{E} \text{ *)} \\
= \; & \mathbf{E}(s)
\end{aligned}$$

for all $s \in [s]_{\sim^b}$.

For condition (B.4), we need to show that each rate of moving from an equivalence class C to some state s in the original system is equal to that when moving from each state C to state $[s]_{\sim^b}$ in the reduced system. Consider IMC $\mathcal{I}$ and its quotient $\mathcal{I}/\sim^b$ in Figure 3.7. The equivalence classes in $\mathcal{I}$ are the following:

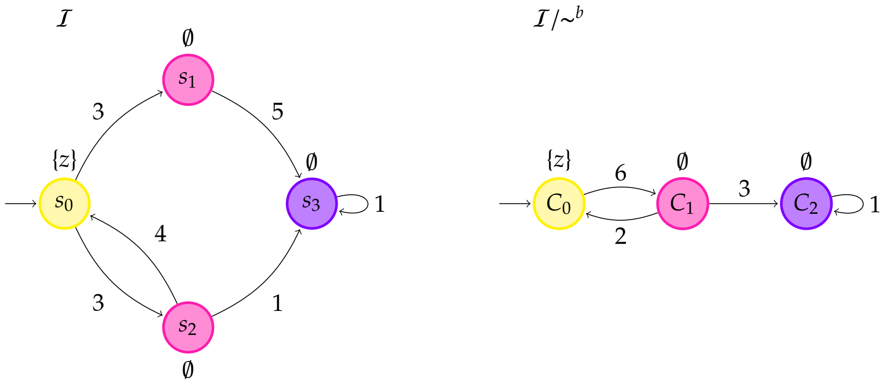

Figure 3.7.: IMC $\mathcal{I}$ and its quotient $\mathcal{I}/\sim^b$, which does not exhibit exact but average rates

$$\begin{aligned} C_0 &= \{ s_0 \}, \\ C_1 &= \{ s_1, s_2 \}, \\ C_2 &= \{ s_3 \}, \end{aligned}$$

In the original system, the rate of moving to state s_3 from equivalence class $\{ s_1, s_2 \}$ is 6, i.e. $\mathbf{R}(\{ s_1, s_2 \}, s_3) = 6$. However, the rate of moving from state $C_1 = \{ s_1, s_2 \}$ is 3 according to Definition 3.2.1 and thus, $\mathbf{R}(C_1, C_2) = 3 \neq 6 = \mathbf{R}(\{ s_1, s_2 \}, s_3)$. This is due to the definition of the quotient, which (only) reflects the *average* rate by which we can move in the original system from some state in equivalence class $\{ s_1, s_2 \}$ to state s_3. As a consequence, there does not exist a state in IMC $\mathcal{I}$ that is pbb to state $C_2 \in S/\sim^b$ and, vice versa, there is no state in IMC $\mathcal{I}/\sim^b$ that is pbb to $s_3 \in S$. A similar argumentation holds for the rates of moving from C_1 to C_0, where we have that $\mathbf{R}(C_1, C_0) = 2 \neq 4 = \mathbf{R}(\{ s_1, s_2 \}, s_0)$. It follows that there exist IMCs that are not pbb to their quotient systems, i.e. $\mathcal{I} \not\sim^b \mathcal{I}/\sim^b$ in general. ■

Note that the IMC in Theorem 3.2.1 is a CTMC, as it does not contain any interactive transitions. Thus, we can conclude that in general a CTMC is not probabilistic backward bisimilar to its quotient.

The following remark, however, shows that under specific circumstances, we can guarantee pbb equivalence of IMCs $\mathcal{I}$ and $\mathcal{I}/\sim^b$.

Remark 3.2.1 (PBB Equivalence of $\mathcal{I}$ and $\mathcal{I}/\sim^b$)
Let $\mathcal{I}$ be an IMC and $\mathcal{I}/\sim^b$ its quotient system. Then, $\mathcal{I} \sim^b \mathcal{I}/\sim^b$ if for all states $s, t \in S$, we have that

$$s \overset{\lambda}{\Longrightarrow} t \in \Longrightarrow \quad \text{implies} \quad |\, [s]_{\sim^b} \,| = |\, [t]_{\sim^b} \,|.$$

Proof Let $\mathcal{I}$ be an IMC and $\mathcal{I}/\sim^b$ its quotient system and let $s, t \in S$ with $s \overset{\lambda}{\Longrightarrow} t \in \Longrightarrow$. First recall that

$$\mathbf{R}(\{ s_1, ..., s_n \}, t) = \mathbf{R}(\{ s_1, ..., s_n \}, t'), \tag{3.1}$$

for all states $t' \in [t]_{\sim^b}$, where $\{ s_1, ..., s_n \} = [s]_{\sim^b}$. Under the assumption that $| [s]_{\sim^b} | = | [t]_{\sim^b} |$, it remains to show that $\mathbf{R}(\{ s_1, ..., s_n \}, t) = \mathbf{R}([s]_{\sim^b}, [t]_{\sim^b})$:

$$\begin{aligned}
& \mathbf{R}([s]_{\sim^b}, [t]_{\sim^b}) \\
& \text{(* by Definition 3.2.1 *)} \\
= \; & \frac{1}{| [s]_{\sim^b} |} \cdot \left(\sum_{s \in [s]_{\sim^b}} \sum_{t \in [t]_{\sim^b}} \mathbf{R}(s, t) \right) \\
& \text{(* since } | [s]_{\sim^b} | = | [t]_{\sim^b} | \text{ *)} \\
= \; & \frac{1}{| [t]_{\sim^b} |} \cdot \left(\sum_{s \in [s]_{\sim^b}} \sum_{t \in [t]_{\sim^b}} \mathbf{R}(s, t) \right) \\
& \text{(* according to Equation 3.1 *)} \\
= \; & \sum_{s \in [s]_{\sim^b}} \mathbf{R}(s, t) \\
& \text{(* by Definition of } \mathbf{R} \text{ *)} \\
= \; & \mathbf{R}(\{ s_1, ..., s_n \}, t),
\end{aligned}$$

for all states $t \in [t]_{\sim^b}$. In this case, condition (B.4) holds and it directly follows that the original system is pbb to its quotient. ∎

Example 3.2.1 illustrates Remark 3.2.1 and Example 3.2.2 shows the concept of quotienting for more complex models, where Remark 3.2.1 is satisfied.

Example 3.2.1 (Non-PBB Equivalence of $\mathcal{I}$ and $\mathcal{I}/\sim^b$)
Consider Figure 3.8, where the right IMC is the probabilistic backward bisimulation quotient system of the left one. Observe that state s_2 reaches the pbb equivalent states s_1 and s_2 via a Markovian transition and the equivalence class of s_0 is a singleton, i.e. we have that $s_0 \overset{2}{\Longrightarrow} s_1$ and $s_0 \overset{2}{\Longrightarrow} s_2$ but $| C_0 | = | [s_0]_{\sim^b} | < | [s_1]_{\sim^b} | = | C_1 |$. Hence, Remark 3.2.1 is not satisfied and the original and its quotient system cannot be pbb. The problem occurs when computing the rate by which the reduced system moves from class $[s_0]_{\sim^b}$ to class $[s_1]_{\sim^b}$:

$$\begin{aligned}
\mathbf{R}([s_0]_{\sim^b}, [s_1]_{\sim^b}) &= \frac{\sum_{s \in [s_0]_{\sim^b}} \sum_{s' \in [s_1]_{\sim^b}} \mathbf{R}(s, s')}{| [s_0]_{\sim^b} |} \\
&= \frac{\mathbf{R}(s_0, s_1) + \mathbf{R}(s_0, s_2)}{1} \\
&= 2 + 2 \\
&= 4 \\
&\neq 2 \\
&= \mathbf{R}(s_0, s_1) \\
&= \mathbf{R}(s_0, s_2).
\end{aligned}$$

As a consequence, $s_1 \not\sim^b C_1 \not\sim^b s_2$ and since there exist no other pbb equivalent candidates for both s_1/s_2 and C_1, it follows immediately that $\mathcal{I} \not\sim^b \mathcal{I}/\sim^b$. On the other hand, consider the Markovian transitions from states s_1 and s_2 that lead to state s_3, where $| [s_1]_{\sim^b} | = 2 >$

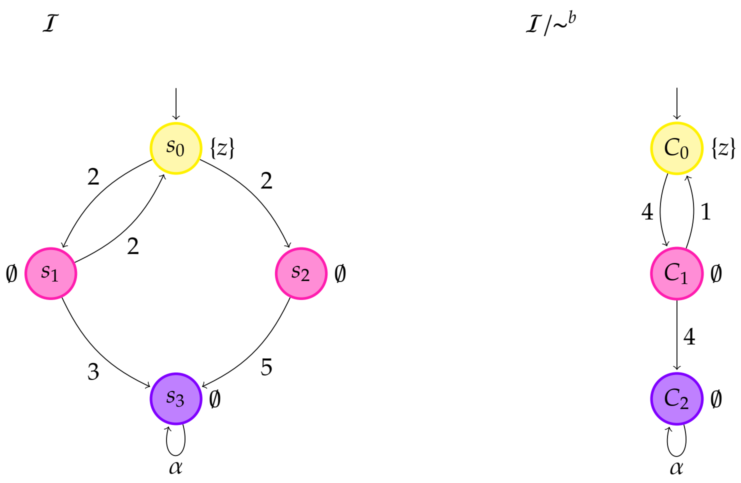

Figure 3.8.: IMC $\mathcal{I}$ (left) and its Quotient System $\mathcal{I}/{\sim^b}$ (right)

$1 = |\, [s_3]_{\sim^b} \,| = |\, C_2 \,|$. In the quotient system, we can reach class $[s_3]_{\sim^b}$ from $[s_1]_{\sim^b}$ via the Markovian 4-transition and it holds that

$$\begin{aligned} \mathbf{R}([s_1]_{\sim^b}, [s_3]_{\sim^b}) &= \frac{\sum_{s \in [s_1]_{\sim^b}} \sum_{s' \in [s_3]_{\sim^b}} \mathbf{R}(s, s')}{|\, [s_1]_{\sim^b} \,|} \\ &= \frac{\mathbf{R}(s_1, s_3) + \mathbf{R}(s_1, s_3)}{2} \\ &= \frac{3+5}{2} \\ &= 4 \end{aligned}$$

while $\mathbf{R}(s_1, s_3) = 3$ and $\mathbf{R}(s_2, s_3) = 5$. Thus, $s_3 \not\sim^b C_2$ and $\mathcal{I} \not\sim^b \mathcal{I}/{\sim^b}$. □

Example 3.2.2 (PBB Equivalence of $\mathcal{I}$ and $\mathcal{I}/{\sim^b}$)
Consider IMC $\mathcal{I}$ and its quotient system according to Definition 3.2.1 given in Figure 3.9, where $Act_i = \{\, \emptyset \,\}$. The equivalence classes under $\sim^b$ in $\mathcal{I}$ are

$$\begin{aligned} C_0 &= \{\, s_0 \,\}, \\ C_1 &= \{\, s_1, s_2, s_3 \,\}, \\ C_3 &= \{\, s_4, s_5, s_6 \,\}, \\ C_4 &= \{\, s_7, s_8, s_9 \,\}. \end{aligned}$$

Each state within equivalence class $[s_1]_{\sim^b}$ in IMC $\mathcal{I}$ is reachable by an interactive α-transition and it directly follows that $s_i \sim^b C_1$ for all $s_i \in [s_1]_{\sim^b}$. Observe that in all other cases, we have that whenever there is a Markovian transition from state s_i to state s_j, $i, j \in \{\, 1, ..., 9 \,\}$, then $|[s_i]_{\sim^b}| = |[s_j]_{\sim^b}|$, thus satisfying Remark 3.2.1 and therefore condition (B.4) of Definition 3.1.2. Recall that a state in the quotient system and the states in the corresponding equivalence class satisfy all requirements of Definition 3.1.2, except for the rate condition (B.4). Hence, it now directly follows that $\mathcal{I} \sim^b \mathcal{I}/{\sim^b}$ and

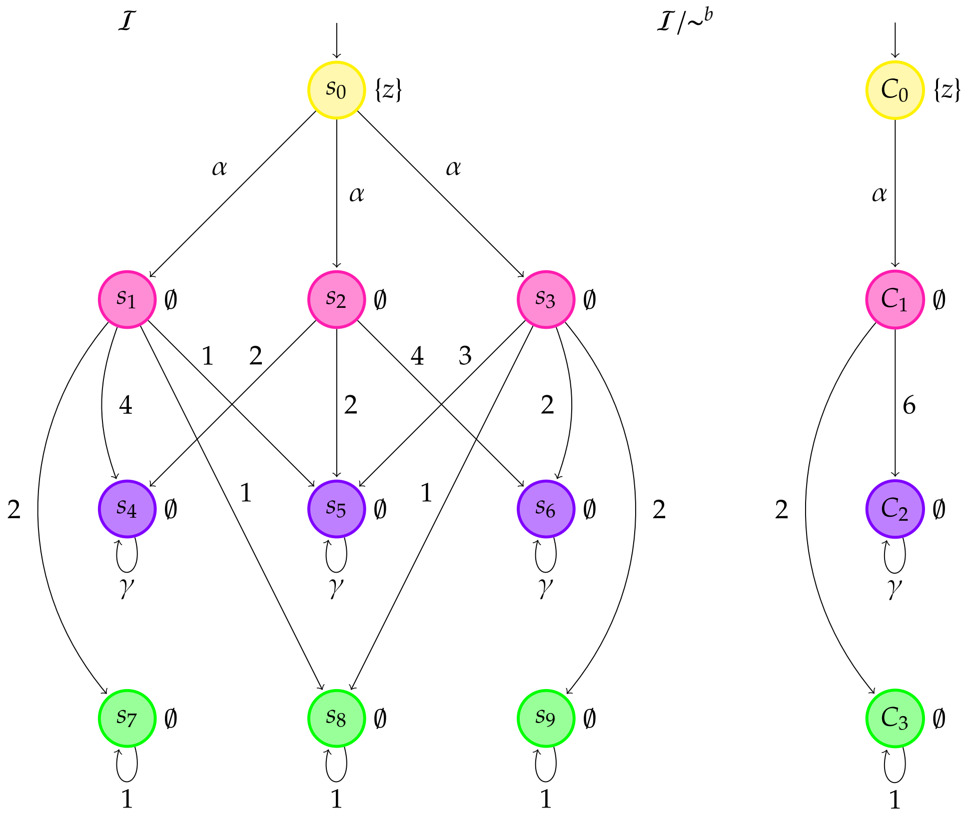

Figure 3.9.: IMC $\mathcal{I}$ (left) and its Quotient System $\mathcal{I}/\sim^b$ (right)

$$\sim^b = \{ (s_0, C_0), (s_1, C_1), (s_2, C_1), (s_3, C_1), (s_4, C_2), (s_5, C_2), (s_6, C_2), (s_7, C_3), (s_8, C_3), (s_9, C_3) \}$$

is a probabilistic backward bisimulation for the pair $(\mathcal{I}, \mathcal{I}/\sim^b)$. □

Quotienting can reduce the original system in its number of states and transitions, which is of course a great advantage, if we want to perform model checking analyses. At first sight, however, there seems to still exist a huge drawback: Quotienting is not an on-the-fly algorithm, i.e. minimization requires the entire state space of the IMC in advance. Assume a huge composed IMC $\mathcal{I}$ of the form

$$\mathcal{I} = \mathcal{I}_1 \,\|_{A_1}\, \mathcal{I}_2 \,\|_{A_2} \dots \|_{A_{n-1}} \mathcal{I}_n,\; n \in \mathbb{N}.$$

Computing the parallel composition of the above n IMCs to analyze the resulting IMC $\mathcal{I}$ is, to put it mildly, problematic. According to Baier and Katoen [2], the parallel composition is the "[...] Cartesian product of the local state spaces S_i of the components [...]" and "[t]he number of states in S is growing [...] *exponentially* in the number of components [...]". Thus, the state space of the composed IMC $\mathcal{I}$ might, and in realistic examples will be too large to be handled and minimization is not applicable. However, each state space of the n components might be of adequate size to efficiently compute the quotients $\mathcal{I}_i/\sim^b$ of $\mathcal{I}_i$ for all $1 \leq i \leq n$.

For that, recall that pbb is a congruence w.r.t. parallel composition (see Theorem 3.1.1), i.e. $\mathcal{I}_1 \sim^b \mathcal{I}_2$ implies $\mathcal{I}_1 \parallel_A \mathcal{I}_3 \sim^b \mathcal{I}_2 \parallel_A \mathcal{I}_3$, where A is a set of synchronization actions, and it follows that, under the assumption that Remark 3.2.1 is satisfied,

$$\mathcal{I}_1 \parallel_A \mathcal{I}_2 \sim^b \mathcal{I}_1/\sim^b \parallel_A \mathcal{I}_2$$

and, furthermore,

$$\mathcal{I}_1/\sim^b \parallel_A \mathcal{I}_2 \sim^b \mathcal{I}_1/\sim^b \parallel_A \mathcal{I}_2/\sim^b.$$

Thus, since $\sim^b$ is transitive, we obtain

$$\mathcal{I}_1 \parallel_A \mathcal{I}_2 \sim^b \mathcal{I}_1/\sim^b \parallel_A \mathcal{I}_2/\sim^b.$$

Now, the fact that pbb is a congruence w.r.t. parallel composition, yields a probably significantly smaller IMC $\mathcal{I}/\sim^b$ of the form

$$\mathcal{I}/\sim^b = \mathcal{I}_1/\sim^b \parallel_{A_1} \mathcal{I}_2/\sim^b \parallel_{A_2} \ldots \parallel_{A_{n-1}} \mathcal{I}_n/\sim^b, n \in \mathbb{N},$$

with the beneficial property that, under the assumption that Remark 3.2.1 holds for all IMCs $\mathcal{I}_1, \ldots, \mathcal{I}_n$, it is pbb to the original, composed IMC $\mathcal{I}$, i.e.

$$\mathcal{I} = \mathcal{I}_1 \parallel_{A_1} \mathcal{I}_2 \parallel_{A_2} \ldots \parallel_{A_{n-1}} \mathcal{I}_n \quad \sim^b \quad \mathcal{I}_1/\sim^b \parallel_{A_1} \mathcal{I}_2/\sim^b \parallel_{A_2} \ldots \parallel_{A_{n-1}} \mathcal{I}_n/\sim^b = \mathcal{I}/\sim^b.$$

Then, property analyses could be (more efficiently) performed on the reduced IMC $\mathcal{I}/\sim^b$.

Example 3.2.3 (Non-PBB Equivalence of $\mathcal{I}_1 \parallel \mathcal{I}_2$ and $\mathcal{I}_1/\sim^b \parallel \mathcal{I}_2/\sim^b$)
Consider the IMCs $\mathcal{I}_1$ and $\mathcal{I}_2$ and their respective quotients $\mathcal{I}_1/\sim^b$ and $\mathcal{I}_2/\sim^b$ given in Figure 3.10. In Figure 3.11 the parallel compositions $\mathcal{I}_1 \parallel_{\{\beta\}} \mathcal{I}_2$ and $\mathcal{I}_1/\sim^b \parallel_{\{\beta\}} \mathcal{I}_2/\sim^b$ can be observed. Note that since $\mathcal{I}_2 = \mathcal{I}_2/\sim^b$, it obviously holds that $\mathcal{I}_2 \sim^b \mathcal{I}_2/\sim^b$. According to Remark 3.2.1, we have that $\mathcal{I}_1 \not\sim^b \mathcal{I}_1/\sim^b$ as $|\,[s_1]_{\sim^b}\,| = 2 \neq 1 = |\,[s_3]_{\sim^b}\,|$ although $s_1 \overset{3}{\Longrightarrow} s_3$.
We will now show that $(\mathcal{I}_1 \parallel_{\{\beta\}} \mathcal{I}_2) \not\sim^b (\mathcal{I}_1/\sim^b \parallel_{\{\beta\}} \mathcal{I}_2/\sim^b)$. To do so, we have to find at least one state $s_1 \in (S_1 \times S_2)$ or $s_2 \in (S_1/\sim^b \times S_2/\sim^b)$ with $s_1 \not\sim^b t_1$ for all $t_1 \in (S_1/\sim^b \times S_2/\sim^b)$ or $s_2 \not\sim^b t_2$ for all $t_2 \in (S_1 \times S_2)$. We choose, for example, $s_2 = (C_2, t_0)$. Since s_2 has an incoming β-transition, it cannot be candidate to be bisimilar to states $(s_0, t_0), (s_0, t_1), (s_1, t_1), (s_2, t_1)$ and (s_3, t_1) as that would violate condition (B.2) of Definition 3.1.2. Due to the different exit rates $\mathbf{E}((s_1, t_0)) = \mathbf{E}((s_2, t_0)) = 10 \neq 7 = \mathbf{E}(s_2)$, it holds that $(s_1, t_0) \not\sim^b s_2 \not\sim^b (s_2, t_0)$. It remains to show that $(s_3, t_0) \not\sim^b s_2$. Observe that states (s_3, t_0) and s_2 are reachable by states (s_1, t_0) and (s_2, t_0) and $(C_1,)$, respectively, via Markovian transitions.

> **Case 1:** $(s_1, t_0) \sim^b (C_1, t_0) \sim^b (s_2, t_0)$. But then, $\mathbf{R}([(s_1, t_0)]_{\sim^b}, s_2) = \mathbf{R}([(s_2, t_0)]_{\sim^b}, s_2) = \mathbf{R}((C_1, t_0), s_2) = 2 \neq 4 = \mathbf{R}((C_1, t_0), (s_3, t_0)) = \mathbf{R}([(s_2, t_0)]_{\sim^b}, (s_3, t_0)) = \mathbf{R}([(s_1, t_0)]_{\sim^b}, (s_3, t_0))$.
>
> **Case 2:** $(s_1, t_0) \not\sim^b (C_1, t_0) \sim^b (s_2, t_0)$. Then, we have that $\mathbf{R}([s_2, t_0], s_2) = 2 \neq 1 = \mathbf{R}([s_2, t_0], (s_3, t_0))$.

A similar argumentation holds for the remaining cases $(s_1, t_0) \sim^b (C_1, t_0) \not\sim^b (s_2, t_0)$ and $(s_1, t_0) \not\sim^b (C_1, t_0) \not\sim^b (s_2, t_0)$. Consequently, we obtain that $s_2 \not\sim^b (s_3, t_0)$, which leads to non-pbb equivalence for the pair $((\mathcal{I}_1 \parallel_{\{\beta\}} \mathcal{I}_2), (\mathcal{I}_1/\sim^b \parallel_{\{\beta\}} \mathcal{I}_2/\sim^b))$. □

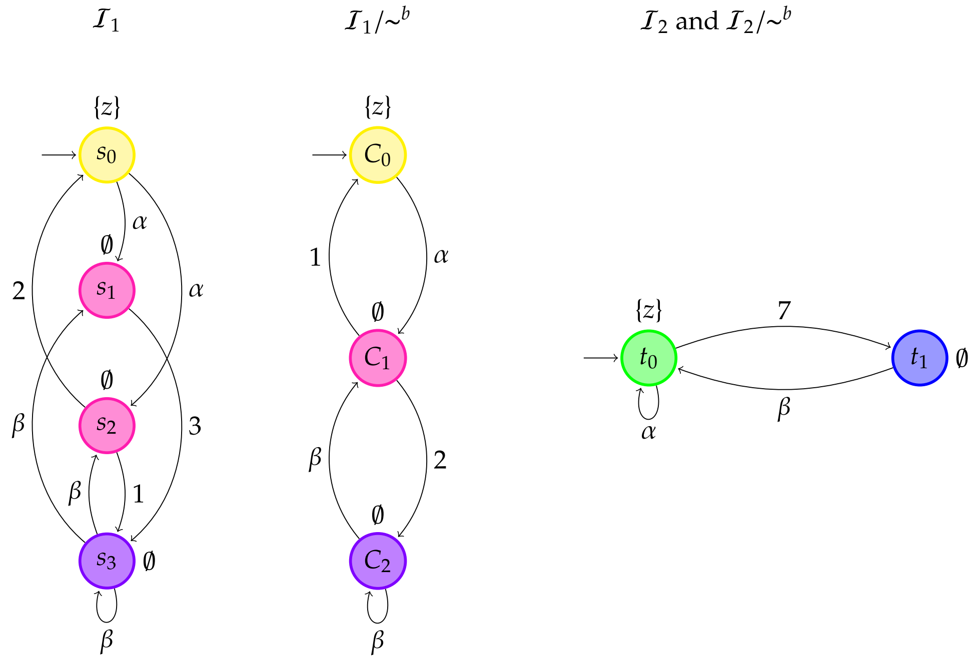

Figure 3.10.: Original IMCs $\mathcal{I}_1$ and $\mathcal{I}_2$ and the corresponding Quotient Systems $\mathcal{I}_1/{\sim^b}$ and $\mathcal{I}_2/{\sim^b}$

Example 3.2.4 (PBB Equivalence of $\mathcal{I}_1 \parallel \mathcal{I}_2$ and $\mathcal{I}_1/{\sim^b} \parallel \mathcal{I}_2/{\sim^b}$)
In Figure 3.12, we are given IMC $\mathcal{I}_3$ and its quotient system $\mathcal{I}_3/{\sim^b}$. For parallel compositions, we will reuse systems $\mathcal{I}_2$ and $\mathcal{I}_2/{\sim^b}$ of Figure 3.10. The results can be found in Figure 3.13. Observe that $\mathcal{I}_3 \sim^b \mathcal{I}_3/\sim^b$, since the original system satisfies Remark 3.2.1. This can be seen as follows. There is a Markovian 3-transition from state s_1 to state s_3 and it holds that $|\,[s_1]_{\sim^b}\,| = 2 = |\,[s_3]\,|_{\sim^b}$. Since $[s_1]_{\sim^b} = \{\, s_1, s_2 \,\}$ and $[s_3]_{\sim^b} = \{\, s_3, s_4 \,\}$, the Remark is satisfied for all remaining Markovian transitions and thus, original and quotient system are pbb. Since probabilistic backward bisimulation is a congruence w.r.t. parallel composition, it directly follows that $\mathcal{I}_2 \parallel_{\{\beta\}} \mathcal{I}_3 \sim^b \mathcal{I}_2/\sim^b \parallel_{\{\beta\}} \mathcal{I}_3/\sim^b$ and we can give a probabilisitic backward bisimulation equivalence

$$
\begin{aligned}
\sim^b \quad = \{ \quad & ((s_0,t_0),(C_0,t_0)), \\
& ((s_1,t_0),(C_1,t_0)),((s_2,t_0),(C_1,t_0)), \\
& ((s_3,t_0),(C_2,t_0)),((s_4,t_0),(C_2,t_0)), \\
& ((s_0,t_1),(C_0,t_1)), \\
& ((s_1,t_1),(C_1,t_1)),((s_2,t_1),(C_1,t_1)), \\
& ((s_3,t_1),(C_2,t_1)),((s_4,t_1),(C_2,t_1)),\ \}.
\end{aligned}
$$

□

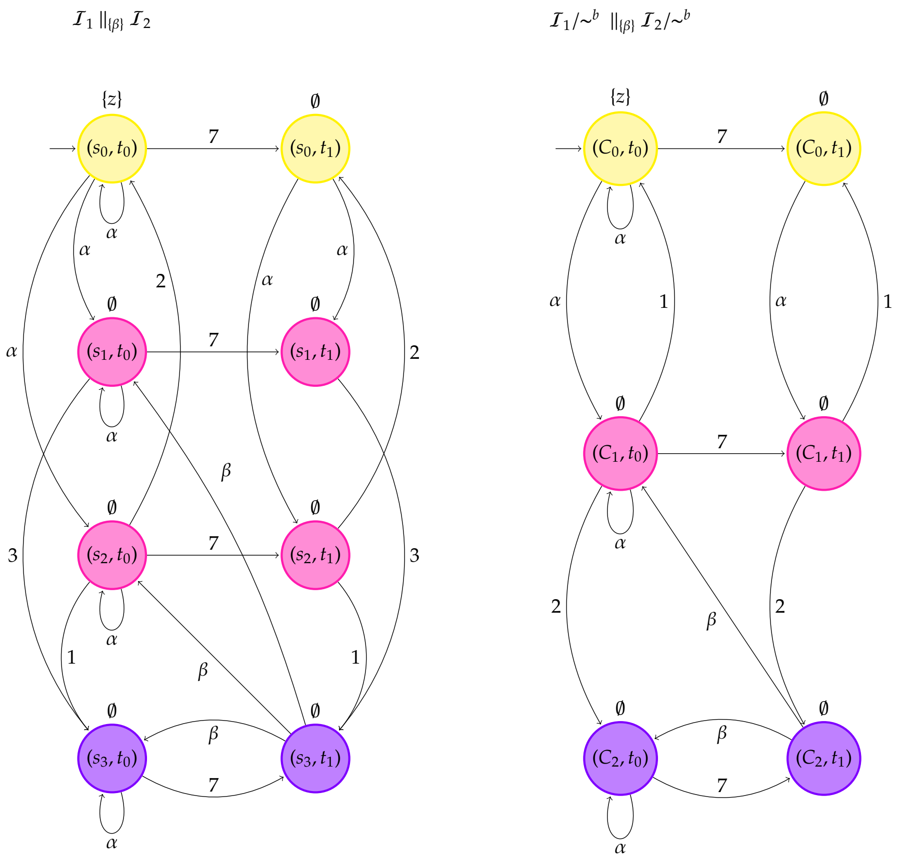

Figure 3.11.: Parallel Compositions of $\mathcal{I}_1$ and $\mathcal{I}_2$ (left), and $\mathcal{I}_1/\sim^b$ and $\mathcal{I}_2/\sim^b$ (right) w.r.t. synchronization action β

3.3. Probabilistic Forward vs. Backward Bisimulation

We will now analyze the differences and similarities between probabilistic forward and backward bisimulation. The definition of the forward variant is based on the approaches by Hermanns and Katoen [13]. However, we adapted it to the case of labeled IMCs.

Definition 3.3.1 (Probabilistic Forward Bisimulation)
Let $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$ be an IMC. An equivalence relation $\mathcal{R} \subseteq S \times S$ is a *probabilistic forward bisimulation*, pfb for short, on $\mathcal{I}$ if for any $(s,t) \in \mathcal{R}$ and equivalence classes $C \in S/\mathcal{R}$ the following holds:

(1) $L(s) = L(t)$,

(2) for any $\alpha \in Act$, $\mathbf{T}(s, \alpha, C) = \mathbf{T}(t, \alpha, C)$, and

(3) if $Post_\tau(s) = \emptyset$ then $\mathbf{R}(s, C) = \mathbf{R}(t, C)$,

where $\mathbf{T}(s, \alpha, C) = 1$ if and only if $\{ s' \in C \mid s \xrightarrow{\alpha} s' \}$ is non-empty and $\mathbf{R}(s, C) = \sum_{s' \in C} \mathbf{R}(s, s')$ is the cumulative rate to go from s to C.

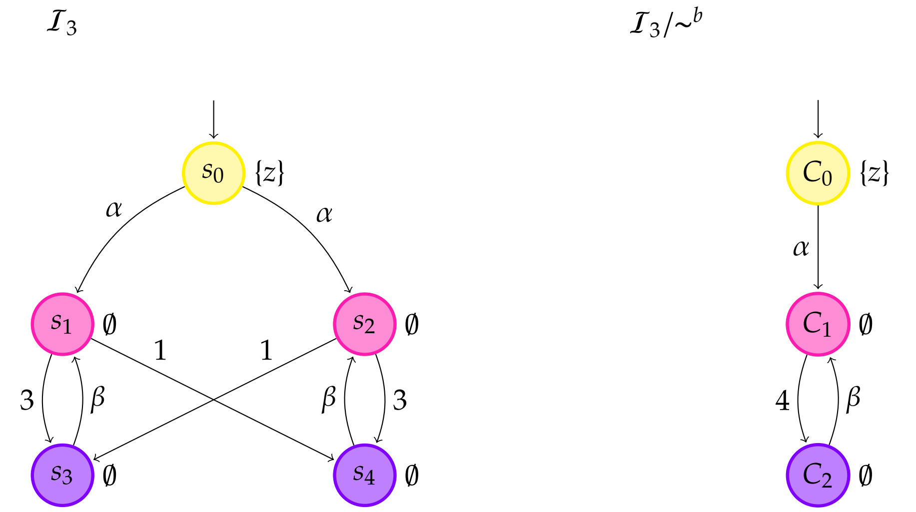

Figure 3.12.: Original IMC $\mathcal{I}_3$ and its Quotient System $\mathcal{I}_3/{\sim^b}$, where $\mathcal{I}_3$ and $\mathcal{I}_3/{\sim^b}$ are pbb

States s and t are *probabilistic forward bisimulation-equivalent or bisimilar (pfb, for short)*, denoted $s \sim t$, if there exists a probabilistic forward bisimulation $\mathcal{R}$ for $\mathcal{I}$ with $(s,t) \in \mathcal{R}$.

Statement (1) of both Definition 3.1.1 and 3.3.1, is the labeling condition, which requires equivalent states to have the same labeling. Requirement (2) of Definition 3.3.1 states that whenever state s can reach a state in some equivalence class $C \in S/\mathcal{R}$ via some action $\alpha \in Act$ then t can reach a state in C via the same action, and vice versa. In addition, it implies that either both states have outgoing interactive transitions or none of them has and thus, no forward-analog to condition (3) of Definition 3.1.1 of the probabilistic backward bisimulation is needed. Furthermore, since requirement (3) of Definition 3.3.1 assures that s and t reach an equivalence class $C \in S/\mathcal{R}$ by the same cumulative rate if they have no outgoing internal interactive transitions, we do not need to explicitly define that if state s does not have any outgoing internal interactive transitions, then the exit rates of s and t must be equal, i.e. $\mathbf{E}(s) = \mathbf{E}(t)$. Hence, statement (5) of Definition 3.1.1 can be omitted in the forward variant.

The following example illustrates the differences between probabilistic forward and backward bisimulation.

Example 3.3.1 (PFB vs. PBB)

Consider Figure 3.14. For the sake of simplicity, assume that all states in both IMCs $\mathcal{I}_f$ and $\mathcal{I}_b$ are equally labeled, except for the initial states labeled by $z \notin AP$. There exists a probabilistic forward bisimulation $\mathcal{R}_f$ for $\mathcal{I}_f$, defined as follows:

$$\mathcal{R}_f = \{ (s_0,s_0),(s_1,s_1),(s_2,s_2),(s_3,s_3),(s_4,s_4),(s_5,s_5),(s_6,s_6), (s_1,s_2),(s_2,s_1),(s_3,s_4),(s_4,s_3),(s_5,s_6),(s_6,s_5) \}.$$

This results in the aggregation of states s_1 and s_2, s_3 and s_4, and s_5 and s_6 into three equivalence classes. There is no pfb state for s_0 and hence, we obtain 4 equivalence classes overall. This

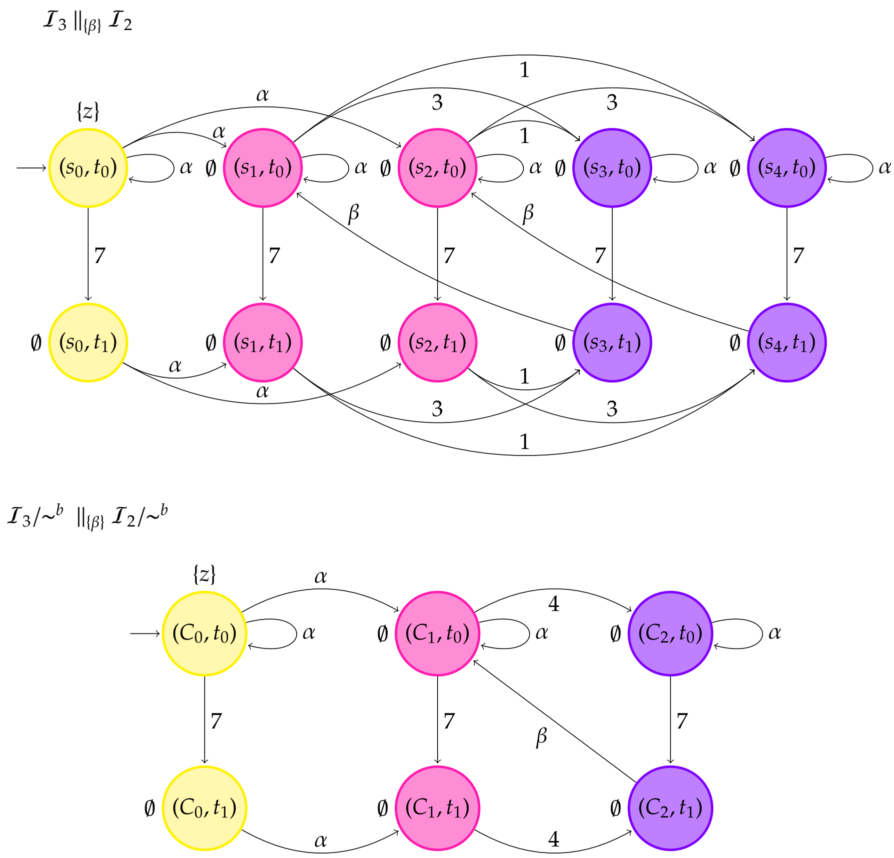

Figure 3.13.: Parallel Compositions of $\mathcal{I}_3$ and $\mathcal{I}_2$ (above), and $\mathcal{I}_3/{\sim^b}$ and $\mathcal{I}_2/{\sim^b}$ (below) w.r.t. synchronization action β

reduces the size of the state space of $\mathcal{I}_f$ from 7 to 4 states. However, applying probabilistic backward bisimulation does not result in any aggregation of states. States s_1 and s_2 are not pbb, since condition (2) of Definition 3.1.1 is violated, i.e. $\mathbf{T}(\{ s_0 \}, \gamma, s_1) = 1 \neq 0 = \mathbf{T}(\{ s_0 \}, \gamma, s_2)$ and $\mathbf{T}(\{ s_0 \}, \alpha, s_1) = 0 \neq 1 = \mathbf{T}(\{ s_0 \}, \alpha, s_2)$. States s_3 and s_4 are not in the same equivalence class under pbb, since $\mathbf{R}(\{ s_1 \}, s_3) = 5 \neq 3 = \mathbf{R}(\{ s_1 \}, s_4)$ and $\mathbf{R}(\{ s_2 \}, s_3) = 5 \neq 3 = \mathbf{R}(\{ s_2 \}, s_4)$, i.e. requirement (4) is not satisfied. Moreover, states s_5 and s_6 cannot be pbb, as $\mathbf{R}(\{ s_3 \}, s_5) = 2 \neq 0 = \mathbf{R}(\{ s_3 \}, s_6)$. In a similar manner, it can be deduced that, except for the identity relation, none of the pairs of states of $\mathcal{I}_f$ are pbb, and thus, probabilistic backward bisimulation does not reduce the state space of IMC $\mathcal{I}_f$.

Considering IMC $\mathcal{I}_b$, there exists a probabilistic backward bisimulation $\mathcal{R}_b$ on $(\mathcal{I}_b, \mathcal{I}_b)$:

$$\begin{aligned}\mathcal{R}_b = \{ & (t_0, t_0), (t_1, t_1), (t_2, t_2), (t_3, t_3), (t_4, t_4), (t_5, t_5), (t_6, t_6), \\ & (t_1, t_2), (t_2, t_1), (t_3, t_4), (t_4, t_3), (t_5, t_6), (t_6, t_5) \}.\end{aligned}$$

According to that, we obtain the following equivalence classes:

$$S/\mathcal{R}_b = \{ \{ t_0 \}, \{ t_1, t_2 \}, \{ t_3, t_4 \}, \{ t_5, t_6 \} \}.$$

Since $\mathbf{T}(t_5, \alpha, \{ t_0 \}) = 1 \neq 0 = \mathbf{T}(t_6, \alpha, \{ t_0 \})$ and, in addition, $\mathbf{T}(t_5, \beta, \{ t_0 \}) = 0 \neq 1 = \mathbf{T}(t_6, \beta, \{ t_0 \})$, states t_5 and t_6 are not probabilistic forward bisimilar, i.e. they violate condition (2) of Definition 3.3.1. Consequently, since t_3 can reach state t_5, whereas t_4 cannot, we have that $\mathbf{R}(t_3, \{ t_5 \}) \neq \mathbf{R}(t_4, \{ t_5 \})$ and it immediately follows that $t_3 \not\sim t_4$. Furthermore, it holds that $t_1 \not\sim t_2$, since $\mathbf{R}(t_1, \{ t_3 \}) = 5 \neq 3 = \mathbf{R}(t_2, \{ t_3 \})$ and $\mathbf{R}(t_1, \{ t_4 \}) = 5 \neq 3 = \mathbf{R}(t_2, \{ t_4 \})$. □

As it can be seen in Example 3.3.1, probabilistic forward and backward bisimulation are incomparable equivalence relations, as their application on the same IMC or pairs of IMCs might result in incomparable equivalence classes and hence, incomparable reductions. Furthermore, according to Hermanns and Katoen [13], for every IMC $\mathcal{I}$, it holds that $\mathcal{I} \sim \mathcal{I}/{\sim}$ and pfb is a congruence w.r.t. parallel composition. Thus, for every two IMCs $\mathcal{I}_1$ and $\mathcal{I}_2$ and set A of synchronization actions, it follows that

$$\mathcal{I}_1 \parallel_A \mathcal{I}_2 \sim \mathcal{I}_1/{\sim} \parallel_A \mathcal{I}_2/{\sim}.$$

For the backward variant, we have that $\mathcal{I} \not\sim^b \mathcal{I}/{\sim^b}$ in general (see Theorem 3.2.1) and the parallel compositions of the original and, resp., reduced systems is pbb solely if Remark 3.2.1 is satisfied.

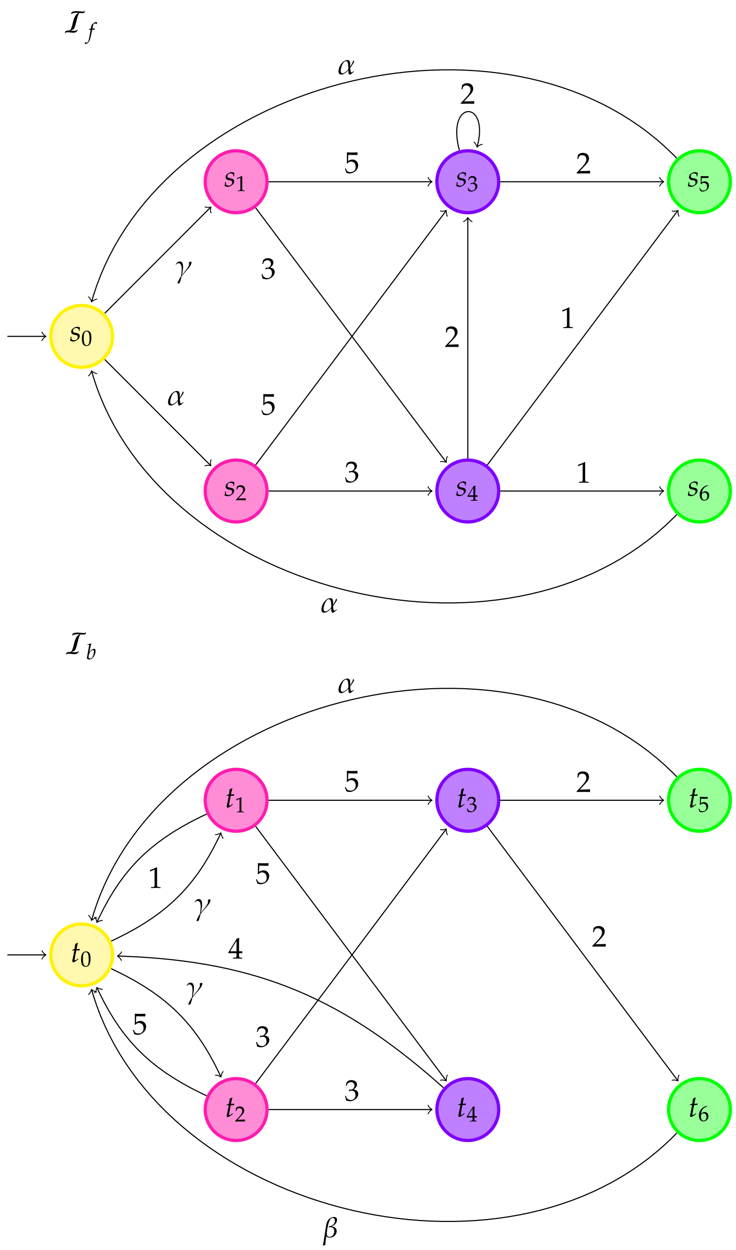

Figure 3.14.: IMCs which can only be reduced by either probabilistic forward or backward bisimulation

4. Property Preservation under $\sim^b$

In this chapter we will discuss the preservation of several timed probabilistic properties under probabilistic backward bisimulation minimization. To that end, note that we cannot expect pbb quotienting to preserve exact probabilities as it computes the average of the original rate values of moving from some equivalence class to another (see Section 3.2). Furthermore, since it requires equivalent states to exhibit the same *past* and basically ignores their future, states within the same class might reach distinct equivalence classes within a single transition and hence, strong linear time behavior will not be preserved for arbitrary pairs of equivalent states. As an example, consider IMC $\mathcal{I}_b$ in Figure 3.14. Here, states t_3 and t_4 are equivalent under $\sim^b$ although t_3 can reach a state inherited in equivalence class $\{ t_5, t_6 \}$ via a Markovian transition, whereas t_4 at least has to visit equivalence classes $\{ t_0 \}$ and $\{ t_1, t_2 \}$ via a Markovian and an interactive γ- transition, until finally reaching state t_3 and moving on to t_5 or t_6.
In the following, we assume the given IMCs to be closed.

4.1. Timed Reachability p^{min}/p^{max}

In this section we shortly introduce the property of interval bounded timed reachability and we strongly follow Neuhäußer [18] and Zhang et al. [23] in the definition. The idea is to compute the probability that, given an initial state s_0 of an arbitrary IMC $\mathcal{I}$, some set $G \subseteq S$ of *goal states* during a given time interval I will be visited. Let $\mathfrak{I}$ be the set of nonempty intervals over the nonnegative reals and let $t \in \mathbb{R}_{\geq 0}$ and $I \in \mathfrak{I}$. Then, we define $I \ominus t = \{ x - t \mid x \in I \wedge x \geq t \}$ and $I \oplus t = \{ x + t \mid x \in I \}$. The set

$$\diamond^I G = \{ \pi \in Paths^{\omega} \mid \exists t \in I.\ \exists s' \in \pi@t.\ s' \in G \}$$

denotes the set of all infinite paths that visit a goal state in G at some point of time in I. The function $p_G^{max}(s, I)$ then denotes the maximum probability induced by $\diamond^I G$ in $\mathcal{I}$ and is defined as follows:

$$p_G^{max}(s, I) = \sup_{D \in GM} Pr^{\omega}_{s,D}(\diamond^I G).$$

According to Neuhäußer [18] and Zhang et al. [23], the function p_G^{max} can be characterized as the least fixed point of the higher order operator Ω, which is defined on measurable functions $F : S \times \mathfrak{I} \mapsto [0, 1]$.

Theorem 4.1.1 ([23] Maximum Timed Reachability as Least Fixed Point)
Let $\mathcal{I}$ be an IMC, $G \subseteq S$ a set of goal states and $I \in \mathfrak{I}$, such that $inf\ I = a$ and $sup\ I = b$. The function $p_G^{max} : S \times \mathfrak{I} \mapsto [0, 1]$ is *the least fixed point* of the higher-order operator $\Omega : (S \times \mathfrak{I} \mapsto [0, 1]) \mapsto (S \times \mathfrak{I} \mapsto [0, 1])$, which is defined as follows:

(1) For Markovian states $s \in MS$:

$$\Omega(F)(s,I) = \begin{cases} \int_0^b E(s)\, e^{-E(s)t} \cdot \sum_{s' \in S} \mathbf{P}(s,s') \cdot F(s', I \ominus t)\, dt & \text{if } s \notin G \\ e^{-E(s)a} + \int_0^a E(s)\, e^{-E(s)t} \cdot \sum_{s' \in S} \mathbf{P}(s,s') \cdot F(s', I \ominus t)\, dt & \text{if } s \in G. \end{cases}$$

(2) For interactive states $s \in IS$:

$$\Omega(F)(s,I) = \begin{cases} 1 & \text{if } s \in G \text{ and } 0 \in I \\ max\{\, F(s', I) \mid s' \in Post_\tau(s)\,\} & \text{otherwise.} \end{cases}$$

Proof For details on the proof, see Neuhäußer [18], Theorem 6.1. ■

The following Theorem shows that quotient system under probabilistic backward bisimulation does **not** preserve the (average) maximum timed reachability probability. (Note that in the Appendix in Table A.1 the reader is provided with the .ma-files for checking the results with the IMC Analyzer [10] implemented by Dennis Guck.)

Theorem 4.1.2 (Timed Reachability is not preserved under $\sim^b$)
Let $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$, be an IMC and $\mathcal{I}/\sim^b = (S/\sim^b, Act, \longrightarrow_{\sim^b}, \Longrightarrow_{\sim^b}, [s_0]_{\sim^b}, AP, L_{\sim^b})$ its quotient system under $\sim^b$. Let $I \subseteq \Im$ be an interval and $G \subseteq S$ the set of goal states, which is closed under $\sim^b$ (i.e. G is the union of zero or more equivalence classes under $\sim^b$), and $G/\sim^b$ its corresponding set of equivalence classes under $S/\sim^b$. Then

$$\frac{\sum_{s \in [s_0]_{\sim^b}} p_G^{max}(s,I)}{|[s_0]_{\sim^b}|} \neq p^{max}_{G/\sim^b}([s_0]_{\sim^b}, I).$$

An analogous statement holds in case of the minimum timed reachability probability p_G^{min} in the original and $p^{min}_{G/\sim^b}$ in the quotient IMC.

Proof To prove our statement, we assume that the timed maximum reachability probability in the quotient is in average equal to that of the original system. First observe that, since $[s_0]_{\sim^b}$ is a singleton by definition (cf. page 4), we have that

$$\frac{\sum_{s \in [s_0]_{\sim^b}} p_G^{max}(s,I)}{|[s_0]_{\sim^b}|} = p_G^{max}(s_0, I) \text{ and, equivalently, } \frac{\sum_{s \in [s_0]_{\sim^b}} p_G^{min}(s,I)}{|[s_0]_{\sim^b}|} = p_G^{min}(s_0, I).$$

We will explicitly compute these values for the closed IMCs given in Figure 4.1, where the left IMC $\mathcal{I}$ is the original model and the right one its probabilistic backward bisimulation quotient $\mathcal{I}/\sim^b$, and show that the resulting values do not coincide. Note that the initial state s_0 of the original system $\mathcal{I}$ has two outgoing internal interactive transitions labeled by the same action τ. As we assume the system to be closed, all internal actions emanating from a state must be uniquely labeled when measuring probabilities (see page 7). Thus, for the following computation, we will rename the outgoing internal actions of state s_0 as follows: $s_0 \stackrel{\tau_1}{\Longrightarrow} s_1$ and $s_0 \stackrel{\tau_2}{\Longrightarrow} s_2$. Furthermore, observe that original and reduced system are probabilistic backward bisimilar and we can give the following probabilistic backward bisimulation for the pair $(\mathcal{I}, \mathcal{I}/\sim^b)$:

$$\sim^b = \{\, (s_0, C_0), (s_1, C_1), (s_2, C_1), (s_3, C_2), (s_4, C_2), (s_5, C_3), (s_6, C_3)\,\}.$$

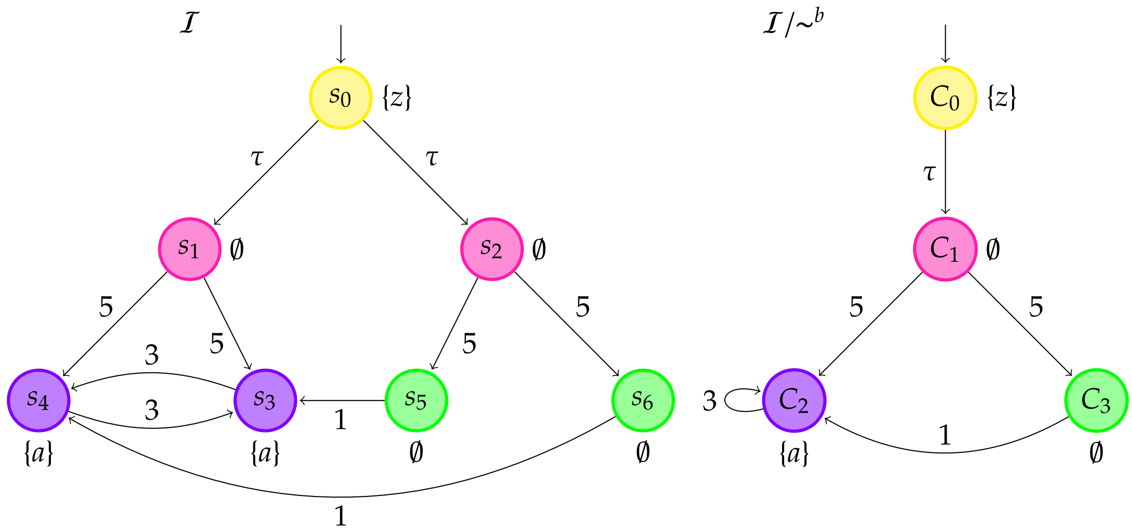

Figure 4.1.: IMC $\mathcal{I}$ (left) and its Quotient System $\mathcal{I}/\sim^b$ (right)

Let $G = \{ s_3, s_4 \}$ and $G/\sim^b = \{ [s_3]_{\sim^b} \} = \{ C_2 \}$ be the set of goal states in the original system and the corresponding set of goal states in the quotient, respectively. We commence by calculating the maximum in the original system of reaching a state in G in the interval $I = [0, 1]$, using the least fixed point computation of Theorem 4.1.1. Note that in the original system the minimum and maximum values must be computed separately, as it exhibits a nondeterministic choice in state s_0.

$$
\begin{aligned}
& p_G^{max}(s_0, [0,1]) \\
& = p_G^{max}(s_0, I) \\
& = max \left\{ p_G^{max}(s', I) \mid s' \in Post_\tau(s_0) \right\} \\
& = max \left\{ p_G^{max}(s_1, I), p_G^{max}(s_2, I) \right\} \\
& = max \left\{ \int_0^b \mathbf{E}(s_1) \cdot e^{-\mathbf{E}(s_1) \cdot t} \sum_{s' \in S} \mathbf{P}(s_1, s') \cdot p_G^{max}(s', I \ominus t) dt, \right. \\
& \qquad \left. \int_0^b \mathbf{E}(s_2) \cdot e^{-\mathbf{E}(s_2) t} \sum_{s' \in S} \mathbf{P}(s_1, s') \cdot p_G^{max}(s', I \ominus t) dt \right\} \\
& \overset{I' = I \ominus t}{=} max \left\{ \int_0^1 10 e^{-10t} \left[\mathbf{P}(s_1, s_3) \cdot p_G^{max}(s_3, I') + \mathbf{P}(s_1, s_4) \cdot p_G^{max}(s_4, I') \right] dt, \right. \\
& \qquad \left. \int_0^1 10 e^{-10t} \left[\mathbf{P}(s_2, s_5) \cdot p_G^{max}(s_5, I') + \mathbf{P}(s_2, s_6) \cdot p_G^{max}(s_6, I') \right] dt \right\}
\end{aligned}
$$

$$= max\left\{ \int_0^1 10e^{-10t} \left[\frac{5}{10} \cdot \left(e^{-\mathbf{E}(s_3)\cdot a} + \int_0^a \mathbf{E}(s_3) \cdot e^{-\mathbf{E}(s_3)\cdot t'} \cdot \sum_{s' \in S} \mathbf{P}(s_3, s') \cdot p_G^{max}(s', I' \ominus t')dt' \right) + \right.\right.$$
$$\left.\left. \frac{5}{10} \cdot \left(e^{-\mathbf{E}(s_4)\cdot a} + \int_0^a \mathbf{E}(s_4) \cdot e^{-\mathbf{E}(s_4)\cdot t'} \cdot \sum_{s' \in S} \mathbf{P}(s_4, s') \cdot p_G^{max}(s', I' \ominus t')dt' \right) \right] dt,\right.$$
$$\int_0^1 10e^{-10t} \left[\frac{5}{10} \cdot \left(\int_0^{1-t} \mathbf{E}(s_5) \cdot e^{-\mathbf{E}(s_5)\cdot t'} \cdot \sum_{s' \in S} \mathbf{P}(s_5, s') \cdot p_G^{max}(s', I' \ominus t')dt' \right) + \right.$$
$$\left.\left. \frac{5}{10} \cdot \left(\int_0^{1-t} \mathbf{E}(s_6) \cdot e^{-\mathbf{E}(s_6)\cdot t'} \cdot \sum_{s' \in S} \mathbf{P}(s_6, s') \cdot p_G^{max}(s', I' \ominus t')dt' \right) \right] dt \right\}$$
$$= max\left\{ \int_0^1 10e^{-10t} \left[\frac{1}{2} \cdot \left(e^{-3\cdot 0} + \int_0^0 3e^{-3t'} \cdot \mathbf{P}(s_3, s_4) \cdot p_G^{max}(s_4, I' \ominus t')dt' \right) + \right.\right.$$
$$\left.\left. \frac{1}{2} \cdot \left(e^{-3\cdot 0} + \int_0^0 3e^{-3t'} \cdot \mathbf{P}(s_4, s_3) \cdot p_G^{max}(s_3, I' \ominus t')dt' \right) \right] dt,\right.$$
$$\int_0^1 10e^{-10t} \left[\frac{1}{2} \cdot \left(\int_0^{1-t} 1e^{-1t'} \cdot \mathbf{P}(s_5, s_3) \cdot p_G^{max}(s_3, I' \ominus t')dt' \right) + \right.$$
$$\left.\left. \frac{1}{2} \cdot \left(\int_0^{1-t} 1e^{-1t'} \cdot \mathbf{P}(s_6, s_4) \cdot p_G^{max}(s_4, I' \ominus t')dt' \right) \right] dt \right\}$$
$$\overset{I''=I'\ominus t'}{=} max\left\{ \int_0^1 10e^{-10t} \left[\frac{1}{2} \cdot (1+0) + \frac{1}{2} \cdot (1+0) \right] dt, \right.$$
$$\int_0^1 10e^{-10t} \left[\frac{1}{2} \cdot \int_0^{1-t} e^{-t'} \cdot \frac{1}{1} \left(e^{-\mathbf{E}(s_3)\cdot a} + \int_0^a \mathbf{E}(s_3) \cdot e^{-\mathbf{E}(s_3)\cdot t''} \cdot \right.\right.$$
$$\left. \sum_{s' \in S} \mathbf{P}(s_3, s') \cdot p_G^{max}(s', I'' \ominus t'')dt'' \right) dt' +$$
$$\frac{1}{2} \cdot \int_0^{1-t} e^{-t'} \cdot \frac{1}{1} \left(e^{-\mathbf{E}(s_4)\cdot a} + \int_0^a \mathbf{E}(s_4) \cdot e^{-\mathbf{E}(s_4)\cdot t''} \cdot \right.$$
$$\left.\left.\left. \sum_{s' \in S} \mathbf{P}(s_4, s') \cdot p_G^{max}(s', I'' \ominus t'')dt'' \right) dt' \right] dt \right\}$$

$$
\begin{aligned}
&= max\left\{ \int_0^1 10e^{-10t}\left[\frac{1}{2}+\frac{1}{2}\right]dt, \right.\\
&\quad \int_0^1 10e^{-10t}\left[\frac{1}{2}\cdot\int_0^{1-t} e^{-t'}\cdot\left(e^{-3\cdot 0}+\int_0^0 3e^{-3t''}\mathbf{P}(s_3,s_4)\cdot p_G^{max}(s_4,I''\ominus t'')dt''\right)dt'+\right.\\
&\quad\quad \left.\left.\frac{1}{2}\cdot\int_0^{1-t} e^{-t'}\left(e^{-3\cdot 0}+\int_0^0 3e^{-3t''}\mathbf{P}(s_4,s_3)\cdot p_G^{max}(s_3,I''\ominus t'')dt''\right)dt'\right]dt\right\}\\
&= max\left\{ \int_0^1 10e^{-10t}dt, \right.\\
&\quad \left.\int_0^1 10e^{-10t}\left[\frac{1}{2}\cdot\int_0^{1-t} e^{-t'}\cdot(1+0)\,dt'+\frac{1}{2}\cdot\int_0^{1-t} e^{-t'}\,(1+0)\,dt'\right]dt\right\}\\
&= max\left\{ \left[-e^{-10t}\right]_0^1, \int_0^1 10e^{-10t}\left[\frac{1}{2}\cdot\int_0^{1-t} e^{-t'}dt'+\frac{1}{2}\cdot\int_0^{1-t} e^{-t'}dt'\right]dt\right\}\\
&= max\left\{ 1-e^{-10}, \int_0^1 10e^{-10t}\left[\frac{1}{2}\cdot\left[-e^{-t'}\right]_0^{1-t}+\frac{1}{2}\cdot\left[-e^{-t'}\right]_0^{1-t}\right]dt\right\}\\
&= max\left\{ 1-e^{-10}, \int_0^1 10e^{-10t}\left[\frac{1}{2}\cdot\left(1-e^{t-1}\right)+\frac{1}{2}\cdot\left(1-e^{t-1}\right)\right]dt\right\}\\
&= max\left\{ 1-e^{-10}, \int_0^1 10e^{-10t}\left[1-e^{t-1}\right]dt\right\}\\
&= max\left\{ 1-e^{-10}, \left[\frac{1}{9}e^{-10t}\left(10e^{t-1}-9\right)\right]_0^1\right\}\\
&= max\left\{ 1-e^{-10}, \frac{1}{9}\left(9+e^{-10}-10e^{-1}\right)\right\}\\
&\approx max\,\{\, 0.99995, 0.59125 \,\}\\
&\approx 0.99995
\end{aligned}
$$

For the minimum transient probability, we have to choose the successor of state s_0 which minimizes the probability. Hence:

$$
\begin{aligned}
p_G^{min}(s_0,[0,1]) &= p_G^{min}(s_0,I)\\
&= min\left\{ 1-e^{-10}, \frac{1}{9}\left(9+e^{-10}-10e^{-1}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\approx \; min\{\, 0.99995, 0.59125 \,\} \\
&\approx \; 0.59125
\end{aligned}
$$

Next, we compute the maximum and minimum timed reachability probabilities in the quotient system for reaching a state in $G/\sim^b$ in the same interval I.

$$
\begin{aligned}
& p^{max}_{G/\sim^b}([s_0]_{\sim^b}, [0,1]) \\
& = p^{max}_{G/\sim^b}(C_0, I) \\
& = max\left\{ p^{max}_{G/\sim^b}(s', I) \mid s' \in Post_\tau(C_0) \right\} \\
& = max\left\{ p^{max}_{G/\sim^b}(C_1, I) \right\} \\
& = \int_0^1 \mathbf{E}(C_1) \cdot e^{-\mathbf{E}(C_1)\cdot t} \cdot \sum_{s' \in S/\sim^b} \mathbf{P}(C_1, s') \cdot p^{max}_{G/\sim^b}(s', I \ominus t) dt \\
& \overset{I'=I\ominus t}{=} \int_0^1 10e^{-10t} \left[\mathbf{P}(C_1, C_2) \cdot p^{max}_{G/\sim^b}(C_2, I') + \mathbf{P}(C_1, C_3) \cdot p^{max}_{G/\sim^b}(C_3, I') \right] dt \\
& = \int_0^1 10e^{-10t} \left[\frac{5}{10} \cdot \left(e^{-\mathbf{E}(C_2)\cdot a} + \int_0^a \mathbf{E}(C_2) \cdot e^{-\mathbf{E}(C_2)\cdot t'} \cdot \sum_{s' \in S/\sim^b} \mathbf{P}(C_2, s') \cdot p^{max}_{G/\sim^b}(s', I' \ominus t') dt' \right) + \right. \\
& \qquad \left. \frac{5}{10} \cdot \left(\int_0^{1-t} \mathbf{E}(C_3) \cdot e^{-\mathbf{E}(C_3)\cdot t'} \cdot \sum_{s' \in S/\sim^b} \mathbf{P}(C_3, s') \cdot p^{max}_{G/\sim^b}(s', I' \ominus t') dt' \right) \right] dt \\
& \overset{I''=I'\ominus t'}{=} \int_0^1 10e^{-10t} \left[\frac{1}{2} \cdot \left(e^{-3\cdot 0} + \int_0^0 3e^{-3t'} \cdot \mathbf{P}(C_2, C_2) \cdot p^{max}_{G/\sim^b}(C_2, I'') dt' \right) + \right. \\
& \qquad \left. \frac{1}{2} \cdot \left(\int_0^{1-t} 1e^{-1t'} \cdot \mathbf{P}(C_3, C_2) \cdot p^{max}_{G/\sim^b}(C_2, I'') dt' \right) \right] dt \\
& = \int_0^1 10e^{-10t} \left[\frac{1}{2} \cdot (1 + 0) + \right. \\
& \qquad \frac{1}{2} \cdot \int_0^{1-t} 1e^{-1t'} \cdot \frac{1}{1} \cdot \left(e^{-\mathbf{E}(C_2)\cdot a} + \int_0^a \mathbf{E}(C_2) \cdot e^{-\mathbf{E}(C_2)\cdot t''} \cdot \right. \\
& \qquad \qquad \left. \left. \sum_{s' \in S/\sim^b} \mathbf{P}(C_2, s') \cdot p^{max}_{G/\sim^b}(s', I'' \ominus t'') dt'' \right) dt' \right] dt
\end{aligned}
$$

$$
\begin{aligned}
&\overset{I'''=I''\ominus t''}{=} \int_0^1 10e^{-10t}\left[\frac{1}{2}+\right.\\
&\qquad \left.\frac{1}{2}\cdot\int_0^{1-t} e^{-t'}\cdot\left(e^{-3\cdot 0}+\int_0^0 3e^{-3t''}\cdot \mathbf{P}(C_2,C_2)\cdot p^{max}_{G/\sim^b}(C_2,I''')dt''\right)dt'\right]dt\\
&= \int_0^1 10e^{-10t}\left[\frac{1}{2}+\frac{1}{2}\cdot\int_0^{1-t} e^{-t'}\cdot(1+0)\,dt'\right]dt\\
&= \int_0^1 10e^{-10t}\left[\frac{1}{2}+\frac{1}{2}\cdot\left[-e^{-t'}\right]_0^{1-t}\right]dt\\
&= \int_0^1 10e^{-10t}\left[\frac{1}{2}+\frac{1}{2}\cdot\left[\left(1-e^{t-1}\right)\right]\right]dt\\
&= \left[\frac{1}{9}e^{-10t}\left(5e^{t-1}-9\right)\right]_0^1\\
&= 1-\frac{4}{9}e^{-10}-\frac{5}{9}e^{-1}
\end{aligned}
$$

(* since for all $s \in S/\sim^b$ we have that $|Post_\tau(s)| \leq 1$ *)

$$
\begin{aligned}
&= p^{min}_{G/\sim^b}(C_0, I)\\
&= p^{min}_{G/\sim^b}([s_0]_{\sim^b}, [0,1])\\
&\approx 0.79560
\end{aligned}
$$

As a result, we obtain

$$
\begin{aligned}
p^{max}_G(s_0,[0,1]) \approx 0.99995 > 0.79560 \approx p^{max}_{G/\sim^b}([s_0]_{\sim^b},[0,1]) \text{ and}\\
p^{min}_G(s_0,[0,1]) \approx 0.59125 < 0.79560 \approx p^{min}_{G/\sim^b}([s_0]_{\sim^b},[0,1]).
\end{aligned}
$$

Thus, it holds that

$$
\begin{aligned}
\frac{\sum_{s\in[s_0]_{\sim^b}} p^{max}_G(s,I)}{|[s_0]_{\sim^b}|} \neq p^{max}_{G/\sim^b}([s_0]_{\sim^b}, I) \text{ and}\\
\frac{\sum_{s\in[s_0]_{\sim^b}} p^{min}_G(s,I)}{|[s_0]_{\sim^b}|} \neq p^{min}_{G/\sim^b}([s_0]_{\sim^b}, I)
\end{aligned}
$$

in general. ■

Note that if the IMC solely consists of Markovian states, i.e. $S = MS$ and the IMC is a CMTC, then the quotient system preserves the *average* transient probability. Thus, the timed reachability probability is in general not preserved under probabilistic backward bisimulation on IMCs due to interactive states. Intuitively, assume that the original or, respectively, reduced system inherits at least one state s with $Post_\tau(s) > 1$, whereas in the corresponding quotient or, respectively, original IMC all interactive states have at most one outgoing interactive transition (cf. Figure 4.1). In this case, every scheduler $D \in GM$ has to *choose* one direct successor state in the original (respectively, reduced) system, whereas in the quotient (respectively, original) model there is *no choice* to be made as every interactive state has a *unique* direct successor. The following Theorem 4.1.3 proves that for CTMCs, the maximum and minimum timed reachability probabilities are preserved. The original proof can be found in [21], which is adapted to our setting and definitions.

Theorem 4.1.3 ([21] Timed Reachability Preservation under $\sim^b$ for CTMCs)
Let $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$, be an IMC with $S = MS$ and $\mathcal{I}/\sim^b = (S/\sim^b, Act, \longrightarrow_{\sim^b}, \Longrightarrow_{\sim^b}, [s_0]_{\sim^b}, AP, L_{\sim^b})$ its quotient system under $\sim^b$. Let $I \subseteq \mathfrak{I}$ be an interval, $G \subseteq S$ the set of goal states, which is closed under $\sim^b$, and $G/\sim^b$ its corresponding set of equivalence classes under $S/\sim^b$. Then

$$\frac{\sum_{s \in [s_0]_{\sim^b}} p_G^{max}(s, I)}{|[s_0]_{\sim^b}|} = p^{max}_{G/\sim^b}([s_0]_{\sim^b}, I).$$

An analogous statement holds in case of the minimum timed reachability probability p_G^{min} and $p^{min}_{G/\sim^b}$.

Proof Let IMCs $\mathcal{I}$ and $\mathcal{I}/\sim^b$, the goal set of states G and interval I be as before with $S = MS$. Define $\tilde{G}$ as the G-corresponding set of equivalence classes in $S/\sim^b$, i.e. $\tilde{G} = G/\sim^b$. In the following, the overline $\sim$ signifies that we are pertaining to values and computations in or elements of the quotient IMC $\mathcal{I}/\sim^b$. Let $f_0, f_1, \ldots$ and $\tilde{f}_0, \tilde{f}_1, \ldots$ be the respective sequences of functions computed to obtain the least fixed points p^{max} for $\mathcal{I}, G$, and I and $\tilde{p}^{max}$ for $\mathcal{I}/\sim^b, \tilde{G}$, and I in Theorem 4.1.1. Our aim is to show that for every interval $J \in \mathfrak{I}$, equivalence class $[s]_{\sim^b} \in S/\sim^b$, and $i \geq 0$, it holds that

$$\tilde{f}_i([s]_{\sim^b}, J) = \frac{\sum_{s \in [s]_{\sim^b}} f_i(s, J)}{|\ [s]_{\sim^b}\ |}.$$

Sproston and Donatelli [21] proved our statement by showing that the average transient probability expressed in Continuous Stochastic Logic is preserved in the quotient CTMC. This can be seen as follows. Recall that the CSL formula $\diamond^I G$ can be reformulated to *true* $\mathcal{U}^I$ G, i.e. $\diamond^I G \equiv_{CSL}$ *true* $\mathcal{U}^I$ G. We will not go into CSL details here, but the reader can find more information in [19, 21, 23]. To identify the set G in terms of CSL, we define a fresh atomic proposition, a say, that is not in AP and set $AP' := AP \cup \{ a \}$ and $L'(s) := L(s) \cup \{ a \}$ if $s \in G$ and $L'_{\sim^b}(s) := L_{\sim^b}(s) \cup \{ a \}$ if $s \in \tilde{G}$ and we obtain two IMCs $\mathcal{I}' = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP', L')$ and $\mathcal{I}'/\sim^b = (S/\sim^b, Act, \longrightarrow_{\sim^b}, \Longrightarrow_{\sim^b}, [s_0]_{\sim^b}, AP', L'_{\sim^b})$ that are isomorphic to their original systems up to the advanced a-labeling of the goal states. Note that adding a label to equivalence classes does not have any impact on the system's maximum probability of reaching a goal state. Then, by Lemma 5.1 in [21], we can establish $\tilde{f}_i([s]_{\sim^b}, J) = \frac{\sum_{s \in [s]_{\sim^b}} f_i(s, J)}{|\ [s]_{\sim^b}\ |}$ for all $i \geq 0$ if $s \in MS$ and since $S = MS$, it follows directly that

$$\frac{\sum_{s \in [s_0]_{\sim^b}} p_G^{max}(s, I)}{|[s_0]_{\sim^b}|} = p^{max}_{G/\sim^b}([s_0]_{\sim^b}, I) = p^{min}_{G/\sim^b}([s_0]_{\sim^b}, I) = \frac{\sum_{s \in [s_0]_{\sim^b}} p_G^{min}(s, I)}{|[s_0]_{\sim^b}|}. \qquad \blacksquare$$

4.2. Minimum/Maximum Expected Time eT^{min}/eT^{max}

The next property we will consider describes the minimum/maximum amount of time needed to reach a goal state $s \in G$ in some IMC $\mathcal{I}$, where the definitions are based on [11,20]. Let $\pi \in Paths^{\omega}$ be an infinite path in $\mathcal{I}$. The (extended) random variable

$$V_G(\pi) = \{ \, t \in \mathbb{R}_{\geq 0} \mid G \cap \pi@t \neq \emptyset \, \}$$

describes the elapsed time on $\pi = s_0 \xrightarrow{\sigma_0, t_0} s_1 \xrightarrow{\sigma_1 t_1} ...$ before a state in G is reached for the first time, where $min(\emptyset) = max(\emptyset) = +\infty$. The function $eT^D(\cdot, \diamond G) : S \mapsto [0, \infty)$ is defined as

$$eT^D(s, \diamond G) = \int_{Paths^{\omega}} V_G(\pi) \; Pr^{\omega}_D(d\pi).$$

It computes the expected time to reach a goal state, given some scheduler $D \in GM$. The *minimum* expected time to reach a goal state from state s of IMC $\mathcal{I}$ is given by the function $eT^{min}(\cdot, \diamond G) : S \mapsto [0, \infty)$, which is defined as

$$eT^{min}(s, \diamond G) = \inf_{D \in GM} \{ \, eT^D(s, \diamond G) \, \}.$$

Analoguously, in case of the *maximum*, the function $eT^{max}(\cdot, \diamond G) : S \mapsto [0, \infty)$ is defined by

$$eT^{max}(s, \diamond G) = \sup_{D \in GM} \{ \, eT^D(s, \diamond G) \, \}.$$

According to Guck et al. [11], the function eT^{min} (resp., eT^{max}) is the least fixed point of the Bellman operator.

Theorem 4.2.1 ([11] Minimum/Maximum Expected Time as Least Fixed Point)
Let $\mathcal{I}$ be an IMC and $G \subseteq S$ a set of goal states. The function $eT^{min} : S \mapsto [0, \infty)$ is the *unique fixed point* of the Bellman operator

$$[L(v)](s) = \begin{cases} \frac{1}{E(s)} + \sum_{s' \in S} \mathbf{P}(s, s') \cdot v(s') & \text{if } s \in MS \backslash G \\ \min\limits_{s \xrightarrow{\alpha} s'} v(s') & \text{if } s \in IS \backslash G \\ 0 & \text{if } s \in G. \end{cases}$$

An analogous statement holds for the maximum eT^{max} except that in an interactive state, the maximal successor will be chosen, i.e.

$$[L(v)](s) = \max_{s \xrightarrow{\alpha} s'} v(s')$$

if $s \in IS \backslash G$. ∎

We will now show that the quotient system under $\sim^b$ does **not** preserve the (average) minimum and maximum expected time. (Note that in the Appendix in Table A.1 the reader is provided with the .ma-files for checking the results with the IMC Analyzer [10] implemented by Dennis Guck.)

Theorem 4.2.2 (Expected Time is not preserved under $\sim^b$)
Let $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$, be an IMC and $\mathcal{I}/\sim^b = (S/\sim^b, Act, \longrightarrow_{\sim^b}, \Longrightarrow_{\sim^b}, [s_0]_{\sim^b}, AP, L_{\sim^b})$ its quotient system under $\sim^b$. Let $G \subseteq S$ be the set of goal states, which is closed under $\sim^b$ (i.e. G is the union of zero or more equivalence classes under $\sim^b$), and $G/\sim^b$ its corresponding set of equivalence classes under $S/\sim^b$. Then

$$\frac{\sum_{s \in [s_0]_{\sim^b}} eT^{min}(s, \diamond G)}{|[s_0]_{\sim^b}|} \neq eT^{min}([s_0]_{\sim^b}, \diamond G/\sim^b),$$

in general. An analogous statement holds in case of the maximum expected time eT^{max} in the original and eT^{max} in the quotient IMC.

Proof To prove our statement, we assume that the minimum/maximum expected time in the quotient is (in average) equal to that of the original system. First note that since $[s_0]_{\sim^b}$ is always a singleton (cf. page 4), it holds that

$$\frac{\sum_{s \in [s_0]_{\sim^b}} eT^{min}(s, \diamond G)}{|[s_0]_{\sim^b}|} = eT^{min}(s_0, \diamond G) \text{ and, similarly, } \frac{\sum_{s \in [s_0]_{\sim^b}} eT^{max}(s, \diamond G)}{|[s_0]_{\sim^b}|} = eT^{max}(s_0, \diamond G).$$

We will explicitly compute these values for the probabilistic backward bisimilar IMCs given in Figure 4.1 and show that the results do not coincide. For that, let $G = \{ s_3, s_4 \}$ and $G/\sim^b = \{ [s_3]_{\sim^b} \} = C_2$. Note that, as we assume the systems to be closed, all internal actions emanating from a state must be uniquely labeled when measuring probabilities (see page 7). Thus, for the following computation, we will rename the outgoing internal actions of state s_0 in IMC $\mathcal{I}$ as follows: $s_0 \overset{\tau_1}{\Longrightarrow} s_1$ and $s_0 \overset{\tau_2}{\Longrightarrow} s_2$.
Following Guck et al. [11], we can compute the minimum (resp. maximum) expected time in both IMCs according to the fixed point computation given in Theorem 4.2.1. In the original system the minimum and maximum values must be computed separately, as it exhibits a nondeterministic choice in state s_0.

$$\begin{aligned}
& eT^{min}(s_0, \diamond G) \\
&= \min_{s_0 \xrightarrow{\alpha} s'} \left\{ eT^{min}(s', \diamond G) \right\} \\
&= \min \left\{ eT^{min}(s_1, \diamond G), eT^{min}(s_2, \diamond G) \right\} \\
&= \min \left\{ \frac{1}{\mathbf{E}(s_1)} + \sum_{s' \in S} \mathbf{P}(s_1, s') \cdot eT^{min}(s', \diamond G), \frac{1}{\mathbf{E}(s_2)} + \sum_{s' \in S} \mathbf{P}(s_2, s') \cdot eT^{min}(s', \diamond G) \right\} \\
&= \min \left\{ \frac{1}{10} + \mathbf{P}(s_1, s_3) \cdot eT^{min}(s_3, \diamond G) + \mathbf{P}(s_1, s_4) \cdot eT^{min}(s_4, \diamond G), \right. \\
&\qquad \left. \frac{1}{10} + \mathbf{P}(s_2, s_5) \cdot eT^{min}(s_5, \diamond G) + \mathbf{P}(s_2, s_6) \cdot eT^{min}(s_6, \diamond G) \right\} \\
&= \min \left\{ \frac{1}{10} + \frac{5}{10} \cdot 0 + \frac{5}{10} \cdot 0, \right. \\
&\qquad \frac{1}{10} + \frac{5}{10} \cdot \left(\frac{1}{\mathbf{E}(s_5)} + \sum_{s' \in S} \mathbf{P}(s_5, s') \cdot eT^{min}(s', \diamond G) \right) +
\end{aligned}$$

$$
\begin{aligned}
& \frac{5}{10} \cdot \left(\frac{1}{\mathbf{E}(s_6)} + \sum_{s' \in S} \mathbf{P}(s_6, s') \cdot eT^{min}(s', \diamond G) \right) \Bigg\} \\
&= min \left\{ \frac{1}{10}, \frac{1}{10} + \frac{1}{2} \cdot \left(\frac{1}{1} + \mathbf{P}(s_5, s_3) \cdot eT^{min}(s_3, \diamond G) \right) + \frac{1}{2} \cdot \left(\frac{1}{1} + \mathbf{P}(s_6, s_4) \cdot eT^{min}(s_4, \diamond G) \right) \right\} \\
&= min \left\{ \frac{1}{10}, \frac{1}{10} + \frac{1}{2} \cdot \left(1 + \frac{1}{1} \cdot 0 \right) + \frac{1}{2} \cdot \left(1 + \frac{1}{1} \cdot 0 \right) \right\} \\
&= min \left\{ \frac{1}{10}, \frac{1}{10} + \frac{1}{2} + \frac{1}{2} \right\} \\
&= min \left\{ \frac{1}{10}, \frac{11}{10} \right\} \\
&= min \{0.1, 1.1\} \\
&= 0.1
\end{aligned}
$$

For the maximum expected time, we have to choose the successor of state s_0 which maximizes the expected time. Hence:

$$
\begin{aligned}
& eT^{max}(s_0, \diamond G) \\
&= max \left\{ \frac{1}{10}, \frac{11}{10} \right\} \\
&= max \{0.1, 1.1\} \\
&= 1.1
\end{aligned}
$$

Next, we compute the maximum and minimum expected time in the quotient system for reaching a state in $G/{\sim^b}$.

$$
\begin{aligned}
& eT^{min}([s_0]_{\sim^b}, \diamond G/{\sim^b}) \\
&= eT^{min}(C_0, \diamond G/{\sim^b}) \\
&= \min_{C_0 \xrightarrow{\alpha} s'} \left\{ eT^{min}(s', \diamond G/{\sim^b}) \right\} \\
&= min \left\{ eT^{min}(C_1, \diamond G/{\sim^b}) \right\} \\
&= eT^{min}(C_1, \diamond G/{\sim^b}) \\
&= \frac{1}{\mathbf{E}(C_1)} + \sum_{s' \in S/{\sim^b}} \mathbf{P}(C_1, s') \cdot eT^{min}(s', \diamond G/{\sim^b}) \\
&= \frac{1}{10} + \mathbf{P}(C_1, C_2) \cdot eT^{min}(C_2, \diamond G/{\sim^b}) + \mathbf{P}(C_1, C_3) \cdot eT^{min}(C_3, \diamond G/{\sim^b}) \\
&= \frac{1}{10} + \frac{5}{10} \cdot 0 + \frac{5}{10} \cdot \left(\frac{1}{\mathbf{E}(C_3)} + \sum_{s' \in S/{\sim^b}} \mathbf{P}(C_3, s') \cdot eT^{min}(s', \diamond G/{\sim^b}) \right) \\
&= \frac{1}{10} + 0 + \frac{1}{2} \cdot \left(\frac{1}{1} + \mathbf{P}(C_3, C_2) \cdot eT^{min}(C_2, \diamond G/{\sim^b}) \right) \\
&= \frac{1}{10} + \frac{1}{2} \cdot \left(1 + \frac{1}{1} \cdot 0 \right)
\end{aligned}
$$

$$= \frac{1}{10} + \frac{1}{2}$$
$$= \frac{6}{10}$$
(* since for all $s \in S/\sim^b$ we have that $|Post_\tau(s)| \leq 1$ *)
$$= eT^{max}(C_0, \diamond G/\sim^b)$$
$$= eT^{max}([s_0]_{\sim^b}, \diamond G/\sim^b)$$
$$= 0.6$$

As a result, we obtain

$$eT^{min}(s_0, \diamond G) = 0.1 < 0.6 = eT^{min}([s_0]_{\sim^b}, \diamond G/\sim^b) \text{ and}$$
$$eT^{max}(s_0, \diamond G) = 1.1 > 0.6 = eT^{max}([s_0]_{\sim^b}, \diamond G/\sim^b).$$

Thus, it holds that

$$\frac{\sum_{s\in[s_0]_{\sim^b}} eT^{min}(s, \diamond G)}{|[s_0]_{\sim^b}|} \neq eT^{min}([s_0]_{\sim^b}, \diamond G/\sim^b) \text{ and}$$
$$\frac{\sum_{s\in[s_0]_{\sim^b}} eT^{max}(s, \diamond G)}{|[s_0]_{\sim^b}|} \neq eT^{max}([s_0]_{\sim^b}, \diamond G/\sim^b)$$

in general. ■

4.3. Minimum Long-Run Average LRA^{min}

The third property refers to the minimum/maximum fraction of time spent in a set of goal states $G \subseteq S$ starting in the initial state s_0 of some IMC $\mathcal{I}$. Definitions again are based on the approaches in [11,20]. The random variable

$$A_G(\pi) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{I}_G(\pi@u)du$$

is the time spent in G on the infinite path π on the long-run, where $\mathbf{I}_G(\pi@u) = 1$ if $\pi@u \cap G \neq \emptyset$ and 0 otherwise. The function $LRA^D(\cdot, G) : S \mapsto [0,1]$ is defined as

$$LRA^D(s, G) = \int_{Paths^\omega} A_G(\pi) Pr^\omega_D(d\pi)$$

and computes the percentage of time spent in G in the long-run starting in state s of IMC $\mathcal{I}$. The *minimum* long-run average then is given by the function $LRA^{min}(\cdot, G) : S \mapsto [0,1]$, which is defined as

$$LRA^{min}(s, G) = \inf_{D\in GM} \{ LRA^D(s, G) \}.$$

Analogously, in case of the *maximum* long-run average, the function $LRA^{max}(\cdot, G) : S \mapsto [0,1]$ is defined as

$$LRA^{max}(s, G) = \sup_{D\in GM} \{ LRA^D(s, G) \}.$$

Note that due to the fact that interactive states are instantaneous, their average residence time is always zero. Thus, we may safely assume w.l.o.g. that the set of goal states solely consists of Markovian states, i.e. $G \subseteq MS$.
Guck et al. [11] presented an algorithm to obtain the minimum/maximum long-run average in IMCs of which we will now provide the basic details. Afterwards, we will compute both minimum and maximum for an IMC and its quotient in Theorem 4.1.2, thus proving by a counterexample that the long-run average is not preserved under probabilistic backward bisimulation. In the following, we will ignore the labeling of the states of the IMC, as the long-run averages focusses on reaching initially specified goal states.

Definition 4.3.1 (Maximal End Component)
Let $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP)$ be an IMC.

- A *sub-IMC* of $\mathcal{I}$ is a pair (S', K), where $S' \subseteq S$ and K is a function that assigns to each state $s \in S'$ a set $\emptyset \neq K(s) \subseteq Act(s)$ of enabled actions, such that for all $\sigma \in K(s)$ the reachable successors are included in S', i.e. $s \xrightarrow{\alpha} t$ or $s \overset{\lambda}{\Longrightarrow} t$ implies $t \in S'$.
- An *end component* is a sub-IMC, whose underlying graph is strongly connected.
- An end component is *maximal* w.r.t. K if it is not contained in any other end component (S'', K).

Definition 4.3.2 (Markov Decision Process)
A *Markov Decision Process* (MDP) is a tuple $\mathcal{M} = (S, Act, \mathbf{P}, s_0)$, where S, Act and s_0 are as for IMCs and $\mathbf{P} : S \times Act \times S \mapsto [0, 1]$ is a *transition probability function*, such that for all $s \in S$ and $\alpha \in Act$, $\sum_{s' \in S} \mathbf{P}(s, \alpha, s') \in \{ 0, 1 \}$.

Definition 4.3.3 (Stochastic Shortest Path Problem)
A non-negative *Stochastic Shortest Path Problem* (SSP Problem) is a tuple $\mathcal{P} = (S, Act, \mathbf{P}, s_0, G, c, g)$, where $(S, Act, \mathbf{P}, s_0)$ is an MDP, $G \subseteq S$ is a set of *goal states*, $c : S \backslash G \times Act \mapsto \mathbb{R}_{\geq 0}$ is a *cost function* and $g : G \mapsto \mathbb{R}_{\geq 0}$ is a *terminal cost function*.

An inifinite sequence $\pi = s_0 \xrightarrow{\alpha_0} s_1 \xrightarrow{\alpha_1} s_2 \xrightarrow{\alpha_2} ...$ is a path in the MDP if for all $i \geq 0$, $s_i \in S$ and $\mathbf{P}(s_i, \alpha_i, s_{i+1}) > 0$. The accumulated cost along path π of reaching a goal state in G is denoted by $C_G(\pi)$ and is computed as the sum over all costs of taking action α_i in state s_i plus the final cost of state $s_k \in G$, where k is the minimal index, such that $s_k \in G$, i.e. $\sum_{i=0}^{k-1} c(s_i, \alpha_i) + g(s_k)$. The minimal expected costs of reaching a goal state in G starting in a state $s \in S$ in some SSP $\mathcal{P}$ will be denoted $cR^{min}(s, \diamond G)$ and is defined as

$$cR^{min}(s, \diamond G) = \inf_D \sum\nolimits_{\pi \in Paths^{\omega}_{abs}} C_G(\pi) \cdot Pr^{\omega,abs}_{s,D}(\pi),$$

where $Paths^{\omega}_{abs}$ denotes the set of (time-abstract) infinite paths in the MDP and $Pr^{\omega,abs}_{s,D}$ the probability measure on sets of paths in the underlying MDP that is induced by scheduler D and initial state s (see Guck et al. [11]. The definition in case of the maximal costs $cR^{max}(s, \diamond G)$ is analogously. The minimal expected cost reachability can be computed by solving a linear programming problem with variables $\{ x_s \}_{s \in S \setminus G}$. The task is to maximize the sum over all x_s-variables subject to the the following constraints for every $s \in S \setminus G$ and $\alpha \in Act$:

$$x_s \leq c(s, \alpha) + \sum_{s' \in S \setminus G} \mathbf{P}(s, \alpha, s') \cdot x_{s'} + \sum_{s' \in G} \mathbf{P}(s, \alpha, s') \cdot g(s').$$

The algorithm to compute the minimum/maximum long-run average in a given IMC $\mathcal{I}$ is organized in three main steps, where for the first a graph-based algorithm is applied (e.g. [5]) and for the last two linear programming problems have to be solved (for more information see [11]).

(1) Determine the maximal end components $\{ \mathcal{I}_1, ..., \mathcal{I}_k \}$ of $\mathcal{I}$.

(2) Determine $LRA^{min}(G)/LRA^{max}(G)$ in maximal end component $\mathcal{I}_j$ for all $j \in \{ 1, ..., k \}$.

(3) Reduce the computation of $LRA^{min}(s_0, G)/LRA^{max}(s_0, G)$ in IMC $\mathcal{I}$ to an SSP Problem.

The second phase requires the transformation of the IMC's maximal end components into MDPs with two cost functions $c_1, c_2 : S \times (Act \cup \{ \perp \}) \mapsto \mathbb{R}_{\geq 0}$, each of them describing the cost for taking some action α in state s along some path π. The *long-run ratio* is the ratio between c_1 and c_2 along the infinite path $\pi = s_0 \xrightarrow{\alpha_0} s_1 \xrightarrow{\alpha_1} s_2 \xrightarrow{\alpha_2} ...$ in the MDP $\mathcal{M}$ and is defined by

$$\mathcal{R}(\pi) = \lim_{n \mapsto \infty} \frac{\sum_{i=0}^{n-1} c_1(s_i, \alpha_i)}{\sum_{j=0}^{n-1} c_2(s_j, \alpha_j)}.$$

The *minimum long-run ratio objective* (and, analogously, the *maximum long-run ratio objective*) for state s of MDP $\mathcal{M}$ is defined by

$$R^{min}(s) = \inf_D \sum_{\pi \in Paths^{\omega}_{abs}} \mathcal{R}(\pi) \cdot Pr^{\omega, abs}_{s,D}(\pi),$$

which can be computed by solving a linear programming problem [8]. According to the following constraints for each state $s \in S$ and $\alpha \in Act$, we obtain a linear system of inequalities

$$x_s \leq c_1(s, \alpha) - k \cdot c_2(s, \alpha) + \sum_{s' \in S} \mathbf{P}(s, \alpha, s') \cdot x_{s'}$$

with real-valued variables k and x_s and where k is to be maximized.
The following Definition 4.3.4 allows the transformation of an IMC into an MDP with two cost functions.

Definition 4.3.4 (Transformation of an IMC into an MDP)
Let $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP)$ be an IMC and $G \subseteq S$ a set of goal states. The induced MDP

is $\mathcal{M}(\mathcal{I}) = (S, Act \cup \{ \perp \}, \mathbf{P}, s_0)$ with cost functions c_1 and c_2, where

$$\mathbf{P}(s, \sigma, s') = \begin{cases} \frac{\mathbf{R}(s,s')}{\mathbf{E}(s)} & \text{if } s \in MS \wedge \sigma = \perp \\ 1 & \text{if } s \in IS \wedge s \xrightarrow{\sigma} s' \\ 0 & \text{otherwise} \end{cases}$$

$$c_1(s, \sigma) = \begin{cases} \frac{1}{\mathbf{E}(s)} & \text{if } s \in MS \cap G \wedge \sigma = \perp \\ 0 & \text{otherwise} \end{cases}$$

$$c_2(s, \sigma) = \begin{cases} \frac{1}{\mathbf{E}(s)} & \text{if } s \in MS \wedge \sigma = \perp \\ 0 & \text{otherwise.} \end{cases}$$

This transformation results in an MDP with two cost functions c_1 and c_2, where c_1 monitors the average time spent in a goal state s, whereas the c_2 keeps track of the average residence time spent in an arbitrary Markovian state. The authors then have shown that the long-run averages of unichain IMCs, "i.e. IMCs that under any stationary deterministic scheduler yield strongly connected graph structures", is equal to the long-run ratio objectives of the corresponding transformed MDP.

Theorem 4.3.1 ([11] $LRA^{min}(s, G)$ equals $R^{min}(s)$)
For unichain IMC $\mathcal{I}$, $LRA^{min}(s, G)$ equals $R^{min}(s)$ in MDP $\mathcal{M}(\mathcal{I})$. ■

The same statement holds in case of the maximum. Since an IMC's maximal end components are unichain IMCs [11], it follows that their long-run averages are equal to the long-run ratio objectives of the corresponding MDPs. Note that in unichain IMCs the long-run averages from any two states s and t are equal, such that we can simply write $LRA^{min}(G)$ instead of $LRA^{min}(s, G)$ and $LRA^{min}(t, G)$.

In the third step, the original problem is reduced to an SSP Problem, i.e. we consider IMCs that are not necessarily unichain. The following Lemma can be analogously reformulated in case of the maximum.

Lemma 4.3.1 ([11] Obtaining LRA^{min} by Decomposition of $\mathcal{I}$)
Let $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP)$ be an IMC, $G \subseteq S$ a set of goal states and $\{ \mathcal{I}_1, ..., \mathcal{I}_k \}$ the set of maximal end components in $\mathcal{I}$ with state spaces $S_1, ..., S_k \subseteq S$. Then

$$LRA^{min}(s_0, G) = \inf_D \sum_{j=1}^{k} LRA_j^{min}(G) \cdot Pr^D(s_0 \models \diamond\Box S_j),$$

where $Pr^D(s_0 \models \diamond\Box S_j)$ is the probability to eventually reach some state in S_j from s_0 under scheduler D and $LRA_j^{min}(G)$ is the long-run average fraction of time spent in $G \cap S_j$ in unichain IMC $\mathcal{I}_j$. ■

Computing the long-run averages can be reduced to a non-negative SSP Problem by replacing each maximal end component $\mathcal{I}_j$ of IMC $\mathcal{I}$ by a new state u_j which is not inherited in S. Let $U = \{ u_1, ..., u_k \}$ be the set of new states for each component $\mathcal{I}_j$, where $U \cap S = \emptyset$.

Definition 4.3.5 ([11] SSP for Long-Run Average)
Let $\mathcal{I}$, $S, G \subseteq S$, $\mathcal{I}_j$ and S_j be as before. The SSP induced by $\mathcal{I}$ for the long-run average fraction of time spent in G is the tuple $\mathcal{P}_{LRA^{min}}(\mathcal{I}) = \left(S \setminus \bigcup_{i=1}^{k} S_i \cup U, Act \cup \{\perp\}, \mathbf{P}', s_0, U, c, g\right)$, where

$$\mathbf{P}'(s,\sigma,s') = \begin{cases} \mathbf{P}(s,\sigma,s') & \text{if } s,s' \in S \setminus \bigcup_{i=1}^{k} S_i \\ \sum_{s' \in S_j} \mathbf{P}(s,\sigma,s') & \text{if } s \in S \setminus \bigcup_{i=1}^{k} S_i \wedge s' = u_j, u_j \in U \\ 1 & \text{if } s = s' = u_i \in U \wedge \sigma = \perp \\ 0 & \text{otherwise.} \end{cases}$$

Here, $\mathbf{P}$ is defined as in Definition 4.3.4. Furthermore, $g(u_i) = LRA_i^{min}(G)$ for $u_i \in U$ and $c(s,\sigma) = 0$ for all s and $\sigma \in Act \cup \{\perp\}$.

The definition is analogous for the maximum long-run average. In the original IMC $\mathcal{I}$, each maximal end component $\mathcal{I}_j$ will be replaced by a fresh single state u_j with a $\perp$-labeled self-loop. The value of the terminal cost function g for each new state u_j is set to the minimum long-run average of $\mathcal{I}_j$, i.e. $g(u_j) = LRA_j^{min}(G)$. The transition probabilities are set similarly to Definition 4.3.4, except for those leading to a state in U. In that case, the probability is the cumulative probability of moving to one of the states in S_j. Note that we are considering closed IMCs, such that for interactive states, all outgoing transitions are uniquely labeled. Hence, $\mathbf{P}'$ is a probability function. The following Theorem 4.3.2 shows the correctness of the given reduction and completes the theoretical fundament of the algorithm to compute the long-run averages in IMCs.

Theorem 4.3.2 ([11] Correctness of the Reduction)
For IMC $\mathcal{I}$ and its induced SSP $\mathcal{P}_{LRA^{min}}$ it holds that

$$LRA^{min}(s,G) = cR^{min}(s, \diamond U),$$

where $cR^{min}(s, \diamond U)$ is the minimal cost reachability of U in SSP $\mathcal{P}_{LRA^{min}}(\mathcal{I})$. ■

The same holds for the maximum long-run average.

We will now show that the quotient system under $\sim^b$ does **not** preserve the (average) minimum and maximum long-run average. (Note that in the Appendix in Table A.2 the reader is provided with the .ma-files for checking the results with the IMC Analyzer [10] implemented by Dennis Guck.)

Theorem 4.3.3 (Long-Run Average is not preserved under $\sim^b$)
Let $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$ be an IMC and $\mathcal{I}/\sim^b = (S/\sim^b, Act, \longrightarrow_{\sim^b}, \Longrightarrow_{\sim^b}, [s_0]_{\sim^b}, AP, L_{\sim^b})$ its quotient system under $\sim^b$. Let $G \subseteq S$ be the set of goal states, which is closed under $\sim^b$ (i.e. G is the union of zero or more equivalence classes under $\sim^b$), and $G/\sim^b$ its corresponding set of equivalence classes under $S/\sim^b$. Then

$$\frac{\sum_{s \in [s_0]_{\sim^b}} LRA^{min}(s,G)}{|[s_0]_{\sim^b}|} \neq LRA^{min}([s_0]_{\sim^b}, G/\sim^b).$$

An analogous statement holds in case of the maximum long-run average LRA^{max} in the original and LRA^{max} in the quotient IMC.

Proof To prove our statement, we assume that the minimum/maximum long-run average in the quotient system is (in average) equal to that of the original system. Recall that, by definition (cf. page 4), equivalence class $[s_0]_{\sim^b}$ is always a singleton and hence, it holds that

$$\frac{\sum_{s\in[s_0]_{\sim^b}} LRA^{min}(s,G)}{|[s_0]_{\sim^b}|} = LRA^{min}(s_0, G) \text{ and, similarly, } \frac{\sum_{s\in[s_0]_{\sim^b}} LRA^{max}(s,G)}{|[s_0]_{\sim^b}|} = LRA^{max}(s_0, G).$$

We will now compute these values according to the algorithm by Guck et al. [11] presented above for the closed IMCs given in Figure 4.2, where the left IMC $\mathcal{I}$ is the original system and the right is its probabilistic backward bisimulation quotient $\mathcal{I}/\sim^b$, and show that the resulting long-run averages do not coincide. Due to the assumption that all internal actions emanating from a state must be uniquely labeled when measuring probabilities (see page 7), we will rename the outgoing internal actions of state s_0 in the original model as follows: $s_0 \overset{\tau_1}{\Longrightarrow} s_1$ and $s_0 \overset{\tau_2}{\Longrightarrow} s_2$. Also note that the original and reduced system are probabilistic backward bisimilar:

$$\sim^b = \{ (s_0, C_0), (s_1, C_1), (s_2, C_1), (s_3, C_2), (s_4, C_2), (s_5, C_3), (s_6, C_3), (s_7, C_4), (s_8, C_4) \}.$$

Let $G = \{ s_3, s_4 \}$ and $G/\sim^b = \{ [s_3]_{\sim^b} \} = \{ C_2 \}$ be the set of goal states in the original system

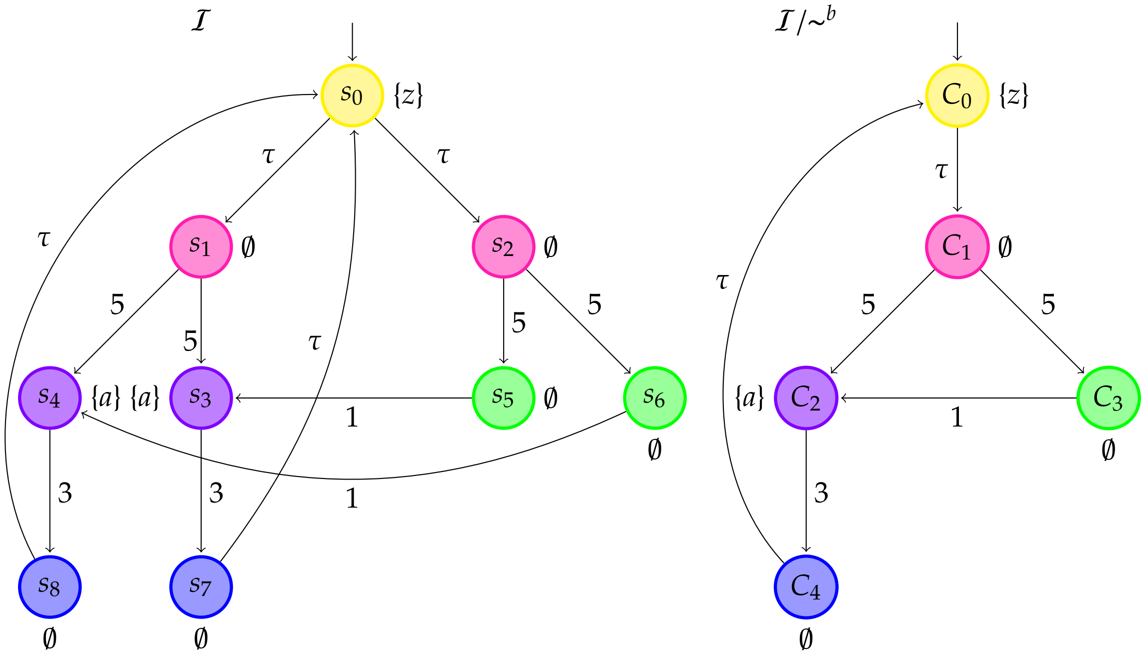

Figure 4.2.: IMC $\mathcal{I}$ (left) and its Quotient System $\mathcal{I}/\sim^b$ (right)

and the corresponding set of goal states in the quotient, respectively. We commence with the computation of the minimum and maximum long-run average in the original system. In the first step, we have to determine the maximal end components (MECs) $\{ \mathcal{I}_1, ..., \mathcal{I}_k \}$ of IMC $\mathcal{I}$. Note that the complete system $\mathcal{I}$ is a MEC of IMC $\mathcal{I}$, since it is a strongly connected graph and is not contained in any other end component. Thus, in phase two, we solely have a single MEC for which we need to determine $LRA^{min}(G)$ and $LRA^{max}(G)$. To do so, the maximal end component $\mathcal{I}$ must be transformed into an MDP $\mathcal{M}(\mathcal{I}) = (S, Act \cup \{ \perp \}, \mathbf{P}, s_0)$ with cost functions c_1 and c_2 according to Definition 4.3.4. For Markovian transitions in the original system, we obtain the transition probability function $\mathbf{P}$ by dividing the rate $\mathbf{R}$ of moving from some state s to some state s' by the total exit rate $\mathbf{E}$ of s. The results can be found in Table 4.1. For the internal action τ_1, we have that $\mathbf{P}(s_0, \tau_1, s_1) = 1$ and

$\mathbf{P}(s, \perp, s')$	$s \in S$ \ $s' \in S$	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
	s_0	0	0	0	0	0	0	0	0	0
	s_1	0	0	0	$\frac{5}{10}$	$\frac{5}{10}$	0	0	0	0
	s_2	0	0	0	0	0	$\frac{5}{10}$	$\frac{5}{10}$	0	0
	s_3	0	0	0	0	0	0	0	$\frac{3}{3}$	0
	s_4	0	0	0	0	0	0	0	0	$\frac{3}{3}$
	s_5	0	0	0	$\frac{1}{1}$	0	0	0	0	0
	s_6	0	0	0	0	$\frac{1}{1}$	0	0	0	0
	s_7	0	0	0	0	0	0	0	0	0
	s_8	0	0	0	0	0	0	0	0	0

Table 4.1.: Transition Probability Function in MDP $\mathcal{M}(\mathcal{I})$ for Markovian Transitions in $\mathcal{I}$

$\mathbf{P}(s, \tau_1, s') = 0$ for all remaining pairs $(s, s') \in (S \times S) \setminus \{ (s_0, s_1) \}$. Similarly, $\mathbf{P}(s_0, \tau_2, s_2) = 1$ and $\mathbf{P}(s, \tau_2, s') = 0$ for all pairs $(s, s') \in (S \times S) \setminus \{ (s_0, s_2) \}$. Considering internal action τ, we obtain the transition probabilities $\mathbf{P}(s_7, \tau, s_0) = \mathbf{P}(s_8, \tau, s_0) = 1$ and $\mathbf{P}(s, \tau, s') = 0$ for all pairs $(s, s') \in (S \times S) \setminus \{ (s_7, s_0), (s_8, s_0) \}$. Cost function c_1 keeps track of the average residence time $\frac{1}{\mathbf{E}(s)}$ spent in a goal state $s \in G = \{ s_3, s_4 \}$. Thus, $c_1(s_3, \perp) = c_1(s_4, \perp) = \frac{1}{3}$ and $c_1(s, \sigma) = 0$ for all $(s, \sigma) \in (S \times Act \cup \{ \perp \}) \setminus \{ (s_3, \perp), (s_4, \perp) \}$. The second cost function c_2 reflects the average residence time spent in a state in S. Table 4.2 provides these values for Markovian transitions. Note that the residence time spent in an interactive state is always zero and thus, we obtain $c_2(s, \sigma) = 0$ for all $s \in S$ and $\sigma \in Act$. Next, our aim is to compute the long-run ratio $R^{min}(s_0)$ and $R^{max}(s_0)$ of MDP $\mathcal{M}(\mathcal{I})$, as they coincide with the minimum and maximum long-run averages of IMC $\mathcal{I}$ (cf. Theorem 4.3.1), respectively. To do so, we have to solve the linear programming problem presented on page 46. For the minimum long-run ratio $R^{min}(s_0)$, we obtain the following system of inequalities:

$$
\begin{aligned}
x_{s_0} &\leq c_1(s_0, \tau_1) - k \cdot c_2(s_0, \tau_1) + \sum\nolimits_{s' \in S} \mathbf{P}(s_0, \tau_1, s') \cdot x_{s'} \\
x_{s_0} &\leq c_1(s_0, \tau_2) - k \cdot c_2(s_0, \tau_2) + \sum\nolimits_{s' \in S} \mathbf{P}(s_0, \tau_2, s') \cdot x_{s'} \\
x_{s_1} &\leq c_1(s_1, \perp) - k \cdot c_2(s_1, \perp) + \sum\nolimits_{s' \in S} \mathbf{P}(s_1, \perp, s') \cdot x_{s'} \\
x_{s_2} &\leq c_1(s_2, \perp) - k \cdot c_2(s_2, \perp) + \sum\nolimits_{s' \in S} \mathbf{P}(s_2, \perp, s') \cdot x_{s'} \\
x_{s_3} &\leq c_1(s_3, \perp) - k \cdot c_2(s_3, \perp) + \sum\nolimits_{s' \in S} \mathbf{P}(s_3, \perp, s') \cdot x_{s'}
\end{aligned}
$$

$c_2(s,\perp)$	$s \in S$ / $s' \in S$	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
	s_0	0	0	0	0	0	0	0	0	0
	s_1	0	0	0	$\frac{1}{10}$	$\frac{1}{10}$	0	0	0	0
	s_2	0	0	0	0	0	$\frac{1}{10}$	$\frac{1}{10}$	0	0
	s_3	0	0	0	0	0	0	0	$\frac{1}{3}$	0
	s_4	0	0	0	0	0	0	0	0	$\frac{1}{3}$
	s_5	0	0	0	$\frac{1}{1}$	0	0	0	0	0
	s_6	0	0	0	0	$\frac{1}{1}$	0	0	0	0
	s_7	0	0	0	0	0	0	0	0	0
	s_8	0	0	0	0	0	0	0	0	0

Table 4.2.: Cost Function c_2 in MDP $\mathcal{M}(\mathcal{I})$ for Markovian Transitions in $\mathcal{I}$

$$
\begin{aligned}
x_{s_4} &\leq c_1(s_4,\perp) - k\cdot c_2(s_4,\perp) + \sum\nolimits_{s'\in S} \mathbf{P}(s_4,\perp,s')\cdot x_{s'}\\
x_{s_5} &\leq c_1(s_5,\perp) - k\cdot c_2(s_5,\perp) + \sum\nolimits_{s'\in S} \mathbf{P}(s_5,\perp,s')\cdot x_{s'}\\
x_{s_6} &\leq c_1(s_6,\perp) - k\cdot c_2(s_6,\perp) + \sum\nolimits_{s'\in S} \mathbf{P}(s_6,\perp,s')\cdot x_{s'}\\
x_{s_7} &\leq c_1(s_7,\tau) - k\cdot c_2(s_7,\tau) + \sum\nolimits_{s'\in S} \mathbf{P}(s_7,\tau,s')\cdot x_{s'}\\
x_{s_8} &\leq c_1(s_8,\tau) - k\cdot c_2(s_8,\tau) + \sum\nolimits_{s'\in S} \mathbf{P}(s_8,\tau,s')\cdot x_{s'}
\end{aligned}
$$

$$
\begin{aligned}
x_{s_0} &\leq 0 - k\cdot 0 + \mathbf{P}(s_0,\tau_1,s_1)\cdot x_{s_1}\\
x_{s_0} &\leq 0 - k\cdot 0 + \mathbf{P}(s_0,\tau_2,s_2)\cdot x_{s_2}\\
x_{s_1} &\leq 0 - k\cdot \frac{1}{10} + \mathbf{P}(s_1,\perp,s_3)\cdot x_{s_3} + \mathbf{P}(s_1,\perp,s_4)\cdot x_{s_4}\\
x_{s_2} &\leq 0 - k\cdot \frac{1}{10} + \mathbf{P}(s_2,\perp,s_5)\cdot x_{s_5} + \mathbf{P}(s_2,\perp,s_6)\cdot x_{s_6}\\
x_{s_3} &\leq \frac{1}{3} - k\cdot \frac{1}{3} + \mathbf{P}(s_3,\perp,s_7)\cdot x_{s_7}\\
x_{s_4} &\leq \frac{1}{3} - k\cdot \frac{1}{3} + \mathbf{P}(s_4,\perp,s_8)\cdot x_{s_8}\\
x_{s_5} &\leq 0 - k\cdot \frac{1}{1} + \mathbf{P}(s_5,\perp,s_3)\cdot x_{s_3}\\
x_{s_6} &\leq 0 - k\cdot \frac{1}{1} + \mathbf{P}(s_6,\perp,s_4)\cdot x_{s_4}\\
x_{s_7} &\leq 0 - k\cdot 0 + \mathbf{P}(s_7,\tau,s_0)\cdot x_{s_0}\\
x_{s_8} &\leq 0 - k\cdot 0 + \mathbf{P}(s_8,\tau,s_0)\cdot x_{s_0}
\end{aligned}
$$

$$x_{s_0} \leq 0 - k\cdot 0 + 1\cdot x_{s_1} \tag{4.1}$$

$$x_{s_0} \leq 0 - k\cdot 0 + 1\cdot x_{s_2} \tag{4.2}$$

$$x_{s_1} \leq 0 - k \cdot \frac{1}{10} + \frac{5}{10} \cdot x_{s_3} + \frac{5}{10} \cdot x_{s_4} \tag{4.3}$$
$$x_{s_2} \leq 0 - k \cdot \frac{1}{10} + \frac{5}{10} \cdot x_{s_5} + \frac{5}{10} \cdot x_{s_6} \tag{4.4}$$
$$x_{s_3} \leq \frac{1}{3} - k \cdot \frac{1}{3} + \frac{3}{3} \cdot x_{s_7} \tag{4.5}$$
$$x_{s_4} \leq \frac{1}{3} - k \cdot \frac{1}{3} + \frac{3}{3} \cdot x_{s_8} \tag{4.6}$$
$$x_{s_5} \leq 0 - k \cdot \frac{1}{1} + \frac{1}{1} \cdot x_{s_3} \tag{4.7}$$
$$x_{s_6} \leq 0 - k \cdot \frac{1}{1} + \frac{1}{1} \cdot x_{s_4} \tag{4.8}$$
$$x_{s_7} \leq 0 - k \cdot 0 + 1 \cdot x_{s_0} \tag{4.9}$$
$$x_{s_8} \leq 0 - k \cdot 0 + 1 \cdot x_{s_0} \tag{4.10}$$

We will not go into details on how to solve this linear programming problem but the interested reader can find more information in Wunderling [22]). However, maximizing real variable k subject to the constraints (4.1) to (4.10) leads to $R^{min}(s_0) \approx 0.23256$ and, according to Theorem 4.3.1, it follows that $LRA^{min}(G) \approx 0.23256$. In the third and last step, the non-negative SSP Problem $\mathcal{P}_{LRA^{min}}(\mathcal{I})$ (see Definition 4.3.3), which is induced by IMC $\mathcal{I}$ (cf. Definition 4.3.5), is to be solved. The induced SSP $\mathcal{P}_{LRA^{min}}(\mathcal{I})$ is obtained from IMC $\mathcal{I}$ by replacing each MEC $\mathcal{I}_i$ in $\mathcal{I}$ by a fresh state $u_i \notin S$ and equipping u_i with a $\perp$-labeled self-loop. Since $\mathcal{I}$ itself is the only MEC, we obtain the SSP $\mathcal{P}_{LRA^{min}}(\mathcal{I})$ given in Figure 4.3, where $S = U$, $U = \{ u \}$, $\mathbf{P}'(u, \perp, u) = 1$, $g(u) = R^{min}(s_0) = LRA^{min}(G) \approx 0.23256$, and $c(u, \perp) = 0$. Now,

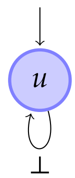

Figure 4.3.: Induced SSP $\mathcal{P}_{LRA^{min}}(\mathcal{I})$

since $LRA^{min}(G) = LRA^{min}(s, G)$ for all states s in the MEC $\mathcal{I}$ (cf. page 47) and according to Theorem 4.3.2, it immediately follows that

$$cR^{min}(s_0, \diamond U) = LRA^{min}(s_0, G) = LRA^{min}(G) \approx 0.23256.$$

For the maximum long-run average, the computations are similar and we obtain $LRA^{max}(G) = R^{max}(s_0) \approx 0.76923$. The SSP $\mathcal{P}_{LRA^{max}}(\mathcal{I})$ is the same as the SSP $\mathcal{P}_{LRA^{min}}(\mathcal{I})$, except that $g(u) = 0.76923$. Using the same argumentation as in case of the minimum, we obtain

$$cR^{max}(s_0, \diamond U) = LRA^{max}(s_0, G) = LRA^{max}(G) \approx 0.76923.$$

It remains to compute the minimum and maximum long-run average for the reduced model $\mathcal{I}/\sim^b$. As for the original system, the quotient $\mathcal{I}/\sim^b$ is a maximal end component and transforming it into an MDP $\mathcal{M}(\mathcal{I}/\sim^b)$ leads to $\mathbf{P}(C_0, \tau, C_1) = \mathbf{P}(C_4, \tau, C_0) = 1$ and $\mathbf{P}(s, \tau, s') = 0$ for all $(s, s') \in (S/\sim^b \times S/\sim^b) \setminus \{ (C_0, C_1), (C_4, C_0) \}$. For the Markovian transitions, the transition probabilities can be found in Table 4.3. Furthermore, we have $c_1(C_2, \bot) = \frac{1}{3}$, $c_1(s, \sigma) = 0$ for

$\mathbf{P}(s, \bot, s')$	$s \in S/\sim^b$ / $s' \in S/\sim^b$	C_0	C_1	C_2	C_3	C_4
	C_0	0	0	0	0	0
	C_1	0	0	$\frac{5}{10}$	$\frac{5}{10}$	0
	C_2	0	0	0	0	$\frac{3}{3}$
	C_3	0	0	$\frac{1}{1}$	0	0
	C_4	0	0	0	0	0

Table 4.3.: Transition Probability Function in MDP $\mathcal{M}(\mathcal{I}/\sim^b)$ for Markovian Transitions in $\mathcal{I}/\sim^b$

all $(s, \sigma) \in (S/\sim^b \times Act \cup \{ \bot \}) \setminus \{ (C_2, \bot) \}$, and $c_2(C_1, \bot) = \frac{1}{10}$, $c_2(C_2, \bot) = \frac{1}{3}$, $c_2(C_3, \bot) = \frac{1}{1}$, and $c_2(s, \sigma) = 0$ for all $(s, \sigma) \in (S/\sim^b \times Act \cup \{ \bot \}) \setminus \{ (C_1, \bot), (C_2, \bot), (C_3, \bot) \}$. Next, we need to compute the long-run ratio $R^{min}([s_0]_{\sim^b})$ and $R^{max}([s_0]_{\sim^b})$, i.e., in case of the minimum, maximize real variable k subject to the following constraints (4.11) - (4.15):

$$
\begin{aligned}
x_{C_0} &\leq c_1(C_0, \tau) - k \cdot c_2(C_0, \tau) + \sum\nolimits_{s' \in S} \mathbf{P}(C_0, \tau, s') \cdot x_{s'} \\
x_{C_1} &\leq c_1(C_1, \bot) - k \cdot c_2(C_1, \bot) + \sum\nolimits_{s' \in S} \mathbf{P}(C_1, \bot, s') \cdot x_{s'} \\
x_{C_2} &\leq c_1(C_2, \bot) - k \cdot c_2(C_2, \bot) + \sum\nolimits_{s' \in S} \mathbf{P}(C_2, \bot, s') \cdot x_{s'} \\
x_{C_3} &\leq c_1(C_3, \bot) - k \cdot c_2(C_3, \bot) + \sum\nolimits_{s' \in S} \mathbf{P}(C_3, \bot, s') \cdot x_{s'} \\
x_{C_4} &\leq c_1(C_4, \tau) - k \cdot c_2(C_4, \tau) + \sum\nolimits_{s' \in S} \mathbf{P}(C_4, \bot, s') \cdot x_{s'}
\end{aligned}
$$

$$
\begin{aligned}
x_{C_0} &\leq 0 - k \cdot 0 + \mathbf{P}(C_0, \tau, C_1) \cdot x_{C_1} \\
x_{C_1} &\leq 0 - k \cdot \frac{1}{10} + \mathbf{P}(C_1, \bot, C_2) \cdot x_{C_2} + \mathbf{P}(C_1, \bot, C_3) \cdot x_{C_3} \\
x_{C_2} &\leq \frac{1}{3} - k \cdot \frac{1}{3} + \mathbf{P}(C_2, \bot, C_4) \cdot x_{C_4} \\
x_{C_3} &\leq 0 - k \cdot \frac{1}{1} + \mathbf{P}(C_3, \bot, C_2) \cdot x_{C_2} \\
x_{C_4} &\leq 0 - k \cdot 0 + \mathbf{P}(C_4, \bot, C_0) \cdot x_{C_0}
\end{aligned}
$$

$$x_{C_0} \leq 0 - k \cdot 0 + 1 \cdot x_{C_1} \tag{4.11}$$

$$x_{C_1} \leq 0 - k \cdot \frac{1}{10} + \frac{5}{10} \cdot x_{C_2} + \frac{5}{10} \cdot x_{C_3} \tag{4.12}$$

$$x_{C_2} \leq \frac{1}{3} - k \cdot \frac{1}{3} + \frac{3}{3} \cdot x_{C_4} \tag{4.13}$$

$$x_{C_3} \leq 0 - k \cdot \frac{1}{1} + \frac{1}{1} \cdot x_{C_2} \tag{4.14}$$

$$x_{C_4} \leq 0 - k \cdot 0 + 1 \cdot x_{C_0} \tag{4.15}$$

As a result, we obtain $R^{min}([s_0]_{\sim^b}) \approx 0.35714$ and furthermore, the induced SSP $\mathcal{P}_{LRA^{min}}(\mathcal{I}/{\sim^b})$ is equivalent to $\mathcal{P}_{LRA^{min}(\mathcal{I})}$, except that $g(u) = R^{min}([s_0]_{\sim^b}) = LRA^{min}(G/\sim^b) \approx 0.35714$. For the maximum, it turns out that $LRA^{max}([s_0]_{\sim^b}, G/\sim^b) = LRA^{max}([s_0]_{\sim^b}, G/\sim^b) \approx 0.35714$. Intuitively, this is due to the fact that, in the reduced system $\mathcal{I}/\sim^b$, there exists no state $s \in S/{\sim^b}$ with $|Post_\tau(s)| > 1$ and thus, for any scheduler $D \in GM$, there is no decision to be made. Following page 47 and Theorem 4.3.2, we have that

$$\begin{aligned} & cR^{min}([s_0]_{\sim^b}, \diamond U) = LRA^{min}([s_0]_{\sim^b}, G/{\sim^b}) = LRA^{min}(G/{\sim^b}) \\ \approx\ & 0.35714 \\ \approx\ & LRA^{max}(G/{\sim^b}) = LRA^{max}([s_0]_{\sim^b}, G/{\sim^b}) = cR^{max}([s_0]_{\sim^b}, \diamond U). \end{aligned}$$

As a result, we obtain

$$LRA^{min}(s_0, G) = 0.23256 < 0.35714 = LRA^{min}([s_0]_{\sim^b}, G/{\sim^b}) \text{ and}$$
$$LRA^{max}(s_0, G) = 0.76923 > 0.35714 = LRA^{max}([s_0]_{\sim^b}, G/{\sim^b}).$$

Thus, it holds that

$$\frac{\sum_{s \in [s_0]_{\sim^b}} LRA^{min}(s,G)}{|[s_0]_{\sim^b}|} \neq LRA^{min}([s_0]_{\sim^b}, G/{\sim^b}) \text{ and}$$
$$\frac{\sum_{s \in [s_0]_{\sim^b}} LRA^{max}(s,G)}{|[s_0]_{\sim^b}|} \neq LRA^{max}([s_0]_{\sim^b}, G/{\sim^b})$$

in general. ∎

4.4. Probabilistic Semi- and Partially Backward Bisimulation

In this chapter, we will shortly discuss two variants of probabilistic backward bisimulation and check, whether they allow for reductions that actually preserve timed probabilistic properties, such as (average) minimum and maximum timed reachability, expected time, and long-run average. The first version replaces the rate condition (4) of Definition 3.1.1 by the forward rate condition (3) of Definition 3.3.1. The second transforms requirement (2) and (3) on interactive transitions of Definition 3.1.1 into statement (2) of Definition 3.3.1.

4.4.1. Forward Condition for Rates

One of the problems when it comes to minimization with respect to probabilistic backward bisimulation is that original and reduced system are in general not bisimilar. This was due to the fact, that the rate condition (B.4) of Definition 3.1.2 is in general not preserved. As an idea to resolve this complication, we will now introduce a semi-backward variant of pbb, where condition (B.4) is transformed into a forward condition on the rates of potentially bisimilar states.

Definition 4.4.1 (Probabilistic Semi-Backward Bisimilar IMCs)
Let $\mathcal{I}_i = (S_i, Act, \longrightarrow_i, \Longrightarrow_i, s_{0,i}, AP, L_i)$, $i = 1, 2$, be two IMCs. A *probabilistic semi-backward bisimulation*, *psbb* for short, for $(\mathcal{I}_1, \mathcal{I}_2)$ is an equivalence relation $\mathcal{R} \subseteq S_1 \times S_2$, such that

(A) $(s_{0,1}, s_{0,2}) \in \mathcal{R}$,

(B) for all $(s_1, s_2) \in \mathcal{R}$ and equivalence classes $C \in (S_1 \uplus S_2)/\mathcal{R}$

  (1) $L(s_1) = L(s_2)$,

  (2) for any $\alpha \in Act$, $\mathbf{T}(C, \alpha, s_1) = \mathbf{T}(C, \alpha, s_2)$,

  (3) $Post_\alpha(s_1) = \emptyset$ if and only if $Post_\alpha(s_2) = \emptyset$, and

  (4) if $Post_\tau(s_1) = \emptyset$ then $\mathbf{R}(s_1, C) = \mathbf{R}(s_2, C)$,

where $(S_1 \uplus S_2)/\mathcal{R}$ denotes the state space $S_1 \uplus S_2$ with respect to $\mathcal{R}$, i.e. the set of all psbb-equivalence classes under $\mathcal{R}$.

$\mathcal{I}_1$ and $\mathcal{I}_2$ are *psbb*, denoted $\mathcal{I}_1 \sim^{sb} \mathcal{I}_2$, if there exists a probabilistic semi-backward bisimulation $\mathcal{R}$ for $(\mathcal{I}_1, \mathcal{I}_2)$.

Note that psbb-equivalent states s_1, s_2 have the same exit rates $\mathbf{E}(s_1) = \mathbf{E}(s_2)$, such that we can omit condition (B.5) of Definition 3.1.2 safely. Next we provide the formal definition of the quotient system of some IMC $\mathcal{I}$ under $\sim^{sb}$.

Definition 4.4.2 (Quotient System under $\sim^{sb}$)
For IMC $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$ the probabilistic semi-backward bisimulation quotient IMC $\mathcal{I}/\sim^{sb}$ is defined as follows.

$$\mathcal{I}/\sim^{sb} = (S/\sim^{sb}, Act, \longrightarrow_{\sim^{sb}}, \Longrightarrow_{\sim^{sb}}, [s_0]_{\sim^{sb}}, AP, L_{\sim^{sb}}),$$

where

- $\longrightarrow_{\sim^{sb}}$ is defined by $\dfrac{s \xrightarrow{\alpha} t}{[s]_{\sim^{sb}} \xrightarrow{\alpha}_{\sim^{sb}} [t]_{\sim^{sb}}}$, $\alpha \in Act$,
- $\Longrightarrow_{\sim^{sb}}$ is defined by

$$\frac{s \stackrel{\lambda_1}{\Longrightarrow} t}{[s]_{\sim^{sb}} \stackrel{\lambda_2}{\Longrightarrow}_{\sim^{sb}} [t]_{\sim^{sb}}}, \text{ with } \lambda_2 = \textstyle\sum_{t \in [t]_{\sim^{sb}}} \mathbf{R}(s, t),\ \lambda_1, \lambda_2 \in \mathbb{R}_{>0},$$

  and

- $L_{\sim^{sb}}([s]_{\sim^{sb}}) = L(s)$.

We will now show that original and reduced system under $\sim^{sb}$ are in general psbb-equivalent.

Theorem 4.4.1 (PSBB Equivalence of $\mathcal{I}$ and $\mathcal{I}/\sim^{sb}$)
Let $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$, be an IMC and $\mathcal{I}/\sim^{sb} = (S/\sim^{sb}, Act, \longrightarrow_{\sim^{sb}}, \Longrightarrow_{\sim^{sb}}, [s_0]_{\sim^{sb}}, AP, L_{\sim^{sb}})$ its quotient system under $\sim^{sb}$. Then

$$\mathcal{I} \sim^{sb} \mathcal{I}/{\sim^{sb}}.$$

Proof Let $\mathcal{I}$ be a IMC and $\mathcal{I}/{\sim^{sb}}$ its psbb quotient system. Let $\mathcal{R} = \{ (s, [s]_{\sim^{sb}}) \mid s \in S) \}$ and $C \in (S \uplus S/{\sim^{sb}})/\mathcal{R}$ be an equivalence class under $\mathcal{R}$, which is defined as $C = \{ t_1, ..., t_n, [t]_{\sim^{sb}} \mid [t]_{\sim^{sb}} \in S/{\sim^{sb}}, t_i \in [t]_{\sim^{sb}}, n = |[t]_{\sim^{sb}}| \}$. By definition of the quotient system under $\sim^{sb}$, it can easily be seen that conditions (A) through (B.3) of Definition 4.4.1 are satisfied by all pairs of states in $\mathcal{R}$. To show statement (B.4), first observe that by definition

$$\mathbf{R}(s, \{ t_1, ..., t_n \}) = \mathbf{R}(s', \{ t_1, ..., t_n \}) \tag{4.16}$$

for all states $s' \in [s]_{\sim^{sb}}$, where $\{ t_1, ..., t_n \} = [t]_{\sim^{sb}}$. It remains to show that $\mathbf{R}(s, \{ t_1, ..., t_n \}) = \mathbf{R}([s]_{\sim^{sb}}, [t]_{\sim^{sb}})$:

$$\begin{aligned}
& \mathbf{R}([s]_{\sim^{sb}}, [t]_{\sim^{sb}}) \\
& \text{(* according to Equation 4.16 and by Definition 4.4.2 *)} \\
= \; & \sum\nolimits_{t \in [t]_{\sim^{sb}}} \mathbf{R}(s, t) \\
& \text{(* by Definition of } \mathbf{R}\text{*)} \\
= \; & \mathbf{R}(s, \{ t_1, ..., t_n \}),
\end{aligned}$$

for all states $s \in [s]_{\sim^{sb}}$. Thus, condition (B.4) holds and it directly follows that the original IMC $\mathcal{I}$ is psbb-equivalent to its quotient model $\mathcal{I}/{\sim^{sb}}$. ■

Observe that, in contrast to the quotient under $\sim^b$, the minimized system under $\sim^{sb}$ preserves *exact* and *not average* rates. Next we provide counterexamples to prove that (average) minimum and maximum timed reachability, expected time, and long-run average are **not** preserved under $\sim^{sb}$. (Note that in the Appendix in Table A.3 the reader is provided with the .ma-files for checking the results with the IMC Analyzer [10] implemented by Dennis Guck.)

Theorem 4.4.2 (Timed Reachability is Not preserved under $\sim^{sb}$)
Let $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$, be an IMC and $\mathcal{I}/{\sim^{sb}} = (S/{\sim^{sb}}, Act, \longrightarrow_{\sim^{sb}}, \Longrightarrow_{\sim^{sb}}, [s_0]_{\sim^{sb}}, AP, L_{\sim^{sb}})$ its quotient system under $\sim^{sb}$. Let $I \subseteq \mathfrak{I}$ be an interval and $G \subseteq S$ the set of goal states, which is closed under $\sim^{sb}$ (i.e. G is the union of zero or more equivalence classes under $\sim^{sb}$), and $G/{\sim^{sb}}$ its corresponding set of equivalence classes under $S/{\sim^{sb}}$. Then

$$p_G^{max}(s, I) \neq p_{G/{\sim^{sb}}}^{max}([s_0]_{\sim^{sb}}, I).$$

An analogous statement holds in case of the minimum timed reachability probability p_G^{min} in the original and $p_{G/{\sim^{sb}}}^{min}$ in the quotient IMC.

Proof We will proceed analogously to Theorem 4.1.2. Observe that, since $[s_0]_{\sim^{sb}}$ is a singleton by definition (cf. page 4), we have that the average minimum and, respectively, maximum timed reachability probability coincides with the minimum and, respectively, maximum timed reachability probability, i.e.

$$\frac{\sum_{s \in [s_0]_{\sim^{sb}}} p_G^{max}(s, I)}{|[s_0]_{\sim^{sb}}|} = p_G^{max}(s_0, I) \text{ and, equivalently, } \frac{\sum_{s \in [s_0]_{\sim^{sb}}} p_G^{min}(s, I)}{|[s_0]_{\sim^{sb}}|} = p_G^{min}(s_0, I).$$

Our computation is again based on the algorithmic approach by Neuhäußer [18] (see Theorem 4.1.1), which will be applied to the IMCs given in Figure 4.4, where the IMC $\mathcal{I}$ on the left is the original model and $\mathcal{I}/{\sim^{sb}}$ is its psbb quotient. Note that original and reduced system are psbb and we can provide the following probabilistic semi-backward bisimulation relation:

$$\sim^{sb} = \{ (s_0, C_0), (s_1, C_1), (s_2, C_1), (s_3, C_2), (s_4, C_3) \}.$$

We will calculate the maximum and minimum timed reachability probabilities for both, original and quotient model, of reaching a state in $G = \{ s_3 \}$ and, respectively, $G/{\sim^{sb}} = \{ [s_3]_{\sim^{sb}} \} = \{ C_2 \}$ in the interval $I = [0, 1]$. However, we will not present all computational details here, but simply the main results:

$$p_G^{max}(s_0, [0,1]) \approx 0.83145 < 0.99752 \approx p_{G/\sim^{sb}}^{max}([s_0]_{\sim^{sb}}, [0,1]) \text{ and}$$
$$p_G^{min}(s_0, [0,1]) \approx 0.83145 > 0.79824 \approx p_{G/\sim^{sb}}^{min}([s_0]_{\sim^{sb}}, [0,1]).$$

Thus, it follows that

$$p_G^{max}(s, I) \neq p_{G/\sim^{sb}}^{max}([s_0]_{\sim^{sb}}, I) \text{ and}$$
$$p_G^{min}(s, I) \neq p_{G/\sim^{sb}}^{min}([s_0]_{\sim^{sb}}, I) \qquad \blacksquare$$

in general.

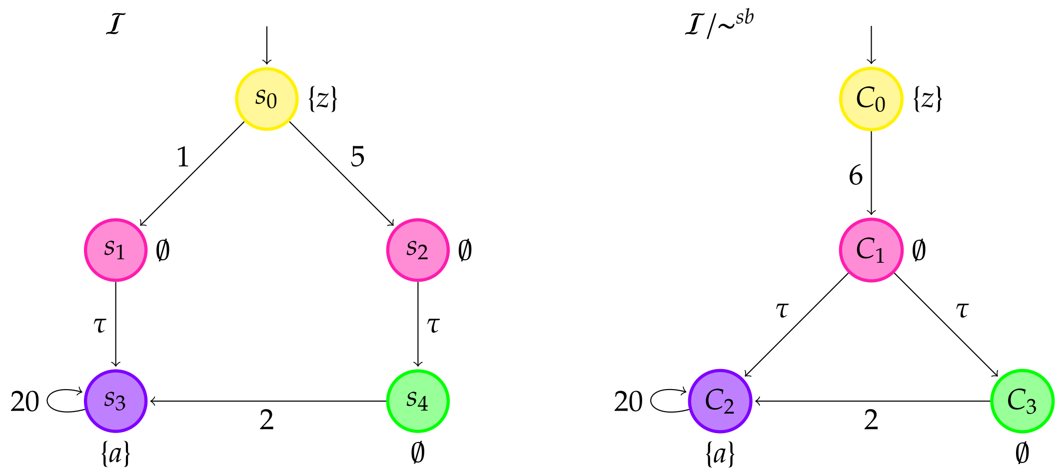

Figure 4.4.: IMC $\mathcal{I}$ (left) and its Quotient System $\mathcal{I}/{\sim^{sb}}$ (right)

(For the expected time analysis, the reader can reuse the .ma-files given in Table A.3 for checking the results with the IMC Analyzer [10] implemented by Dennis Guck.)

Theorem 4.4.3 (Expected Time is not preserved under $\sim^{sb}$)
Let $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$, be an IMC and $\mathcal{I}/\sim^{sb} = (S/\sim^{sb}, Act, \longrightarrow_{\sim^{sb}}, \Longrightarrow_{\sim^{sb}}, [s_0]_{\sim^{sb}}, AP, L_{\sim^{sb}})$ its quotient system under $\sim^{sb}$. Let $G \subseteq S$ be the set of goal states, which is closed under $\sim^{sb}$ (i.e. G is the union of zero or more equivalence classes under $\sim^{sb}$), and $G/{\sim^{sb}}$ its corresponding set of equivalence classes under $S/{\sim^{sb}}$. Then

$$eT^{min}(s, \diamond G) \neq eT^{min}([s_0]_{\sim^{sb}}, \diamond G/{\sim^{sb}}),$$

in general. An analogous statement holds in case of the maximum expected time eT^{max} in the original and eT^{max} in the quotient IMC.

Proof First observe that the average minimum and, respectively, maximum expected time coincides with the minimum and, respectively, maximum expected time, since $[s_0]_{\sim^{sb}}$ is a singleton by definition (cf. page 4), i.e.

$$\frac{\sum_{s\in[s_0]_{\sim sb}} eT^{max}(s,\diamond G)}{|[s_0]_{\sim sb}|} = eT^{max}(s_0, \diamond G) \text{ and, equivalently, } \frac{\sum_{s\in[s_0]_{\sim sb}} eT^{min}(s,\diamond G)}{|[s_0]_{\sim sb}|} = eT^{min}(s_0, \diamond G).$$

Our proof is similar to that of Theorem 4.2.2. According to the fixed point computation given in Theorem 4.2.1 (cf. Guck et al. [11]), we will calculate the minimum and maximum expected time for the psbb-equivalent IMCs $\mathcal{I}$ and $\mathcal{I}/{\sim^{sb}}$ given in Figure 4.4, where $G = \{ s_3 \}$ and $G/{\sim^{sb}} = \{ C_2 \}$. We will not provide the computational details here, but solely the main results:

$$eT^{min}(s_0, \diamond G) = 0.58333 > 0.16667 = eT^{min}([s_0]_{\sim^{sb}}, \diamond G/{\sim^{sb}}) \text{ and}$$
$$eT^{max}(s_0, \diamond G) = 0.58333 < 0.66667 = eT^{max}([s_0]_{\sim^{sb}}, \diamond G/{\sim^{sb}}).$$

Thus, it holds that

$$eT^{min}(s, \diamond G) \neq eT^{min}([s_0]_{\sim^{sb}}, \diamond G/{\sim^{sb}}) \text{ and}$$
$$eT^{max}(s, \diamond G) \neq eT^{max}([s_0]_{\sim^{sb}}, \diamond G/{\sim^{sb}})$$

in general. ∎

(The following results can be checked using the IMC Analyzer [10] implemented by Dennis Guck and the .ma-files provided in Table A.4.)

Theorem 4.4.4 (Long-Run Average is not preserved under $\sim^{sb}$)
Let $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$, be an IMC and $\mathcal{I}/{\sim^{sb}} = (S/{\sim^{sb}}, Act, \longrightarrow_{\sim^{sb}}, \Longrightarrow_{\sim^{sb}}, [s_0]_{\sim^{sb}}, AP, L_{\sim^{sb}})$ its quotient system under $\sim^{sb}$. Let $G \subseteq S$ be the set of goal states, which is closed under $\sim^{sb}$ (i.e. G is the union of zero or more equivalence classes under $\sim^{sb}$), and $G/{\sim^{sb}}$ its corresponding set of equivalence classes under $S/{\sim^{sb}}$. Then

$$LRA^{min}(s, G) \neq LRA^{min}([s_0]_{\sim^{sb}}, G/{\sim^{sb}}).$$

An analogous statement holds in case of the maximum long-run average LRA^{max} in the original and LRA^{max} in the quotient IMC.

Proof First observe that the average minimum and, respectively, maximum long-run average coincide with the minimum and, respectively, maximum long-run average, since $[s_0]_{\sim^{sb}}$ is a singleton by definition (cf. page 4), i.e.

$$\frac{\sum_{s\in[s_0]_{\sim sb}} LRA^{max}(s,G)}{|[s_0]_{\sim sb}|} = LRA^{max}(s_0, G) \text{ and, equivalently, } \frac{\sum_{s\in[s_0]_{\sim sb}} LRA^{min}(s,G)}{|[s_0]_{\sim sb}|} = LRA^{min}(s_0, G).$$

The following proof is similar to that of Theorem 4.3.3. We will compute the minimum and maximum long-run averages in the original IMC $\mathcal{I}$ and its probabilistic semi-backward bisimulation quotient $\mathcal{I}/\sim^{sb}$ shown in Figure 4.5 by applying the algorithm provided in Section 4.3 (cf. Guck et al. [11]). For that, let $G = \{ s_3 \}$ and $G/\sim^{sb} = \{ [s_3]_{\sim^{sb}} \} = \{ C_2 \}$ and observe that $\mathcal{I}$ and $\mathcal{I}/\sim^{sb}$ are psbb:

$$\sim^{sb} = \{ (s_0, C_0), (s_1, C_1), (s_2, C_1), (s_3, C_2), (s_4, C_3) \}.$$

Again, we will not provide the computational details, but simply the main results:

$$LRA^{min}(s_0, G) = 0.07895 > 0.06977 = LRA^{min}([s_0]_{\sim^{sb}}, G/\sim^{sb}) \text{ and}$$
$$LRA^{max}(s_0, G) = 0.07895 < 0.23077 = LRA^{max}([s_0]_{\sim^{sb}}, G/\sim^{sb}).$$

Thus, it holds that

$$LRA^{min}(s, G) \neq LRA^{min}([s_0]_{\sim^{sb}}, G/\sim^{sb}) \text{ and}$$
$$LRA^{max}(s, G) \neq LRA^{max}([s_0]_{\sim^{sb}}, G/\sim^{sb})$$ ■

in general.

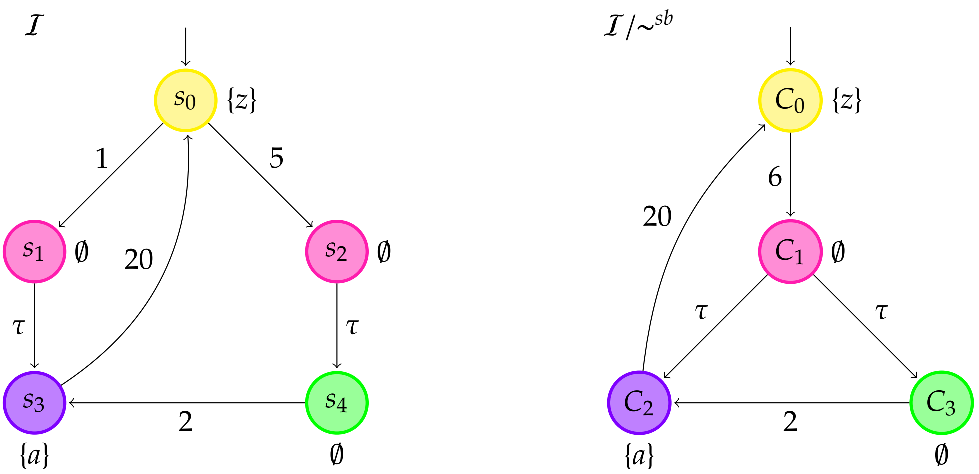

Figure 4.5.: IMC $\mathcal{I}$ (left) and its Quotient System $\mathcal{I}/\sim^{sb}$ (right)

We have shown that under probabilistic semi-backward bisimulation, original and reduced model are always psbb-equivalent but still, none of the examined timed probabilistic properties are preserved if minimization under $\sim^{sb}$ is applied.

4.4.2. Forward Condition for Interactive Transitions

Unfortunately, the problem that pbb does not preserve several timed probabilistic properties is not solved by introducing a forward variant of the rate condition of Definition 3.1.2. Another possibility could be to replace the requirements imposed on interactive transitions in Definition 3.1.2 by a the forward version (2) of Definition 3.3.1 for probabilistic forward bisimulation. As we have seen in both cases, psbb and pbb, the mergence of (at least) two equivalent states of the original system into a single new one in the reduced model is problematic. For example, consider Figure 4.5 of the previous section. In the original IMC $\mathcal{I}$, each state s_1 and s_2 has exactly *one* outgoing interactive transition. In those settings, any scheduler $D \in GM$ can choose nothing but to execute the unique internal interactive transition emanating from s_1 and s_2, respectively. However, these two states are equivalent under psbb and hence, represented by a single new state C_1 in the reduced system $\mathcal{I}/\sim^{sb}$ with the slight but crucial difference, that C_1 now has *two* outgoing τ-transitions, as s_1 and s_2 could reach $different$ equivalence classes in the original systems. An analogous exemplary setting for the probabilistic backward bisimulation equivalence relation can be found in Figure 4.2 for states s_1, s_2 and C_1. The issue in both definitions, psbb and pbb, is that the conditions imposed on interactive transitions do not keep track of *where* interactive transitions lead to but solely whether they *exist*. Thus, in the following, we will try to overcome the property preservation problem by redefining statements (B.2) and (B.3) of Definition 3.1.2.

Definition 4.4.3 (Probabilistic Partially Backward Bisimilar IMCs)
Let $\mathcal{I}_i = (S_i, Act, \longrightarrow_i, \Longrightarrow_i, s_{0,i}, AP, L_i)$, $i = 1, 2$, be two IMCs. A *probabilistic partially backward bisimulation*, *ppbb* for short, for $(\mathcal{I}_1, \mathcal{I}_2)$ is an equivalence relation $\mathcal{R} \subseteq S_1 \times S_2$, such that

(A) $(s_{0,1}, s_{0,2}) \in \mathcal{R}$,

(B) for all $(s_1, s_2) \in \mathcal{R}$ and equivalence classes $C \in (S_1 \uplus S_2)/\mathcal{R}$

  (1) $L(s_1) = L(s_2)$,

  (2) for any $\alpha \in Act$, $\mathbf{T}(s_1, \alpha, C) = \mathbf{T}(s_2, \alpha, C)$,

  (3) if $Post_\tau(C) = \emptyset$ then $\mathbf{R}(C, s_1) = \mathbf{R}(C, s_2)$, and

  (4) if $Post_\tau(s_1) = Post_\tau(s_2) = \emptyset$ then $\mathbf{E}(s_1) = \mathbf{E}(s_2)$,

where $(S_1 \uplus S_2)/\mathcal{R}$ denotes the state space $S_1 \uplus S_2$ with respect to $\mathcal{R}$, i.e. the set of all ppbb-equivalence classes under $\mathcal{R}$, and $\mathbf{T}(s, \alpha, C) = 1$ if and only if $\{ s' \in C \mid s \xrightarrow{\alpha} s' \}$ is non-empty.

$\mathcal{I}_1$ and $\mathcal{I}_2$ are *ppbb*, denoted $\mathcal{I}_1 \sim^{pb} \mathcal{I}_2$, if there exists a probabilistic partially backward bisimulation $\mathcal{R}$ for $(\mathcal{I}_1, \mathcal{I}_2)$.

Observe that condition (B.3) of Definition 3.1.2 is trivially fulfilled whenever requirement (B.2) of Definition 4.4.3 is satisfied and thus, it does not have to be explicitly stated in the Definition of ppbb. Our next task is to formally define the quotient system under ppbb.

Definition 4.4.4 (Quotient System under $\sim^{pb}$)
For IMC $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$ the probabilistic backward bisimulation quotient IMC $\mathcal{I}/\sim^{pb}$ is defined as follows.

$$\mathcal{I}/\sim^{pb} = (S/\sim^{pb}, Act, \longrightarrow_{\sim^{pb}}, \Longrightarrow_{\sim^{pb}}, [s_0]_{\sim^{pb}}, AP, L_{\sim^{pb}}),$$

where

- $\longrightarrow_{\sim pb}$ is defined by $\dfrac{s \xrightarrow{\alpha} t}{[s]_{\sim pb} \xrightarrow{\alpha}_{\sim pb} [t]_{\sim pb}}$, $\alpha \in Act$,
- $\Longrightarrow_{\sim pb}$ is defined by

$$\frac{s \overset{\lambda_1}{\Longrightarrow} t}{[s]_{\sim pb} \overset{\lambda_2}{\Longrightarrow}_{\sim pb} [t]_{\sim pb}}, \text{with } \lambda_2 = \frac{\sum_{s\in[s]_{\sim pb}} \sum_{t\in[t]_{\sim pb}} \mathbf{R}(s,t)}{\mid [s]_{\sim pb} \mid}, \lambda_1, \lambda_2 \in \mathbb{R}_{>0},$$

and

- $L_{\sim pb}([s]_{\sim pb}) = L(s)$.

The following Theorem 4.4.5 shows that original and reduced system under $\sim^{pb}$ are in general **not** ppbb.

Theorem 4.4.5 (Non-PPBB Equivalence of $\mathcal{I}$ and $\mathcal{I}/\sim^{pb}$)
Let $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$, be an IMC and $\mathcal{I}/\sim^{pb} = (S/\sim^{pb}, Act, \longrightarrow_{\sim pb}, \Longrightarrow_{\sim pb}, [s_0]_{\sim pb}, AP, L_{\sim pb})$ its quotient system under $\sim^{pb}$. Then

$$\mathcal{I} \sim^{pb} \mathcal{I}/\sim^{pb}.$$

Proof Let $\mathcal{I}$ be an IMC and $\mathcal{I}/\sim^{pb}$ is ppbb quotient system. As shown in the proof of Theorem 3.2.1, the rate condition (B.3) of Definition 4.4.3 (statement (B.4) in Definition 3.1.2) will not be preserved and thus, we can directly conclude that $\mathcal{I} \not\sim^{pb} \mathcal{I}/\sim^{pb}$ in general. ■

Last but not least, we will show that minimization under $\sim^{pb}$ does not preserve the (average) minimum and maximum timed reachability probability, expected time, and long-run average.

Theorem 4.4.6 (Timed Reachability is Not preserved under $\sim^{pb}$)
Let $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$, be an IMC and $\mathcal{I}/\sim^{pb} = (S/\sim^{pb}, Act, \longrightarrow_{\sim pb}, \Longrightarrow_{\sim pb}, [s_0]_{\sim pb}, AP, L_{\sim pb})$ its quotient system under $\sim^{pb}$. Let $I \subseteq \mathfrak{I}$ be an interval and $G \subseteq S$ the set of goal states, which is closed under $\sim^{pb}$ (i.e. G is the union of zero or more equivalence classes under $\sim^{pb}$), and $G/\sim^{pb}$ its corresponding set of equivalence classes under $S/\sim^{pb}$. Then

$$p^{max}_{G}(s, I) \neq p^{max}_{G/\sim^{pb}}([s_0]_{\sim pb}, I).$$

An analogous statement holds in case of the minimum timed reachability probability p^{min}_{G} in the original and $p^{min}_{G/\sim^{pb}}$ in the quotient IMC.

Proof Consider Figure 4.1 and observe, that the ppbb quotient system of the original IMC $\mathcal{I}$ (left) is isomorphic to that of the pbb quotient $\mathcal{I}/\sim^{b}$. This can be seen as follows. Note that the quotients under $\sim^{pb}$ (cf. Definition 4.4.4) and $\sim^{b}$ (cf. Definition 3.2.1) are constructed analogously. Furthermore, we have that $s_1 \sim^{pb} s_2$, $s_3 \sim^{pb} s_4$, and $s_5 \sim^{pb} s_6$, since they solely have outgoing Markovian transitions and since $s_1 \sim^{b} s_2$, $s_3 \sim^{b} s_4$, and $s_5 \sim^{b} s_6$ (cf. Theorem 4.1.2). As the definition of ppbb does not require equivalent states to be reachable via the same interactive transitions from the same equivalence class, we can directly conclude that $s_1 \sim^{pb} C_1 \sim^{pb} s_2$, $s_3 \sim^{pb} C_2 \sim^{pb} s_4$, and $s_5 \sim^{pb} C_3 \sim^{pb} s_6$. For the initial states s_0 and C_0, observe that all outgoing transitions are labeled by τ and lead to a state contained within equivalence class $\{ s_1, s_2, C_1 \}$. Thus, it holds that $\mathbf{T}(s_0, \tau, \{ s_1, s_2, C_0 \}) = 1 = \mathbf{T}(C_0, \tau, \{ s_1, s_2, C_0 \})$, which fulfills condition (B.2). As neither of them is reachable via or has an emanating Markovian transition, and since $L(s_1) = \{ z \} = L(C_0)$, it follows that $(s_0, C_0) \in \sim^{pb}$ and we obtain the probabilistic partially backward bisimulation equivalence relation

$$\sim^{pb} = \{ (s_0, C_0), (s_1, C_1), (s_2, C_1), (s_3, C_2), (s_4, C_2), (s_5, C_3), (s_6, C_3) \}.$$

Hence, we can conclude that $\mathcal{I} \sim^{pb} \mathcal{I}/{\sim^b}$. Theorem 4.1.2 provides all computational details to calculate the minimum and maximum timed reachability probabilities for both models and since the resulting values do not coincide, we have shown that

$$p_G^{max}(s, I) \neq p_{G/\sim^{pb}}^{max}([s_0]_{\sim^{pb}}, I) \text{ and}$$
$$p_G^{min}(s, I) \neq p_{G/\sim^{pb}}^{min}([s_0]_{\sim^{pb}}, I)$$

in general. ∎

Theorem 4.4.7 (Expected Time is not preserved under $\sim^{pb}$)
Let $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$, be an IMC and $\mathcal{I}/\sim^{pb} = (S/\sim^{pb}, Act, \longrightarrow_{\sim^{pb}}, \Longrightarrow_{\sim^{pb}}, [s_0]_{\sim^{pb}}, AP, L_{\sim^{pb}})$ its quotient system under $\sim^{pb}$. Let $G \subseteq S$ be the set of goal states, which is closed under $\sim^{pb}$ (i.e. G is the union of zero or more equivalence classes under $\sim^{pb}$), and $G/{\sim^{pb}}$ its corresponding set of equivalence classes under $S/{\sim^{pb}}$. Then

$$eT^{min}(s, \diamond G) \neq eT^{min}([s_0]_{\sim^{pb}}, \diamond G/{\sim^{pb}}),$$

in general. An analogous statement holds in case of the maximum expected time eT^{max} in the original and eT^{max} in the quotient IMC.

Proof Consider the ppbb-equivalent IMCs $\mathcal{I}$ and $\mathcal{I}/{\sim^b}$ in Figure 4.1. Theorem 4.2.2 discusses the computation of the minimum and maximum expected time for both original and reduced system in detail and since the resulting values do not coincide, we can directly conclude that

$$eT^{min}(s, \diamond G) \neq eT^{min}([s_0]_{\sim^{pb}}, \diamond G/{\sim^{pb}}) \text{ and}$$
$$eT^{max}(s, \diamond G) \neq eT^{max}([s_0]_{\sim^{pb}}, \diamond G/{\sim^{pb}})$$

in general. ∎

Theorem 4.4.8 (Long-Run Average is not preserved under $\sim^{pb}$)
Let $\mathcal{I} = (S, Act, \longrightarrow, \Longrightarrow, s_0, AP, L)$, be an IMC and $\mathcal{I}/\sim^{pb} = (S/\sim^{pb}, Act, \longrightarrow_{\sim^{pb}}, \Longrightarrow_{\sim^{pb}}, [s_0]_{\sim^{pb}}, AP, L_{\sim^{pb}})$ its quotient system under $\sim^{pb}$. Let $G \subseteq S$ be the set of goal states, which is closed under $\sim^{pb}$ (i.e. G is the union of zero or more equivalence classes under $\sim^{pb}$), and $G/{\sim^{pb}}$ its corresponding set of equivalence classes under $S/{\sim^{pb}}$. Then

$$LRA^{min}(s, G) \neq LRA^{min}([s_0]_{\sim^{pb}}, G/{\sim^{pb}}).$$

An analogous statement holds in case of the maximum long-run average LRA^{max} in the original and LRA^{max} in the quotient IMC.

PROOF Observe that by applying a similar argumentation as in Theorem 4.4.6, it can be deduced that IMCs $\mathcal{I}$ and $\mathcal{I}/\sim^{b}$ in Figure 4.2 are probabilistic partially backward bisimilar and we can give the following ppbb equivalence relation

$$\sim^{pb} = \{ (s_0, C_0), (s_1, C_1), (s_2, C_1), (s_3, C_2), (s_4, C_2), (s_5, C_3), (s_6, C_3), (s_7, C_4), (s_8, C_4) \}.$$

Theorem 4.3.3 explicitly computes the minimum and maximum long-run averages for both models and since the resulting values do not coincide, we can immediately conclude that

$$LRA^{min}(s, G) \neq LRA^{min}([s_0]_{\sim^{pb}}, G/\sim^{pb}) \text{ and}$$
$$LRA^{max}(s, G) \neq LRA^{max}([s_0]_{\sim^{pb}}, G/\sim^{pb})$$

in general. ■

To conclude this chapter, we unfortunately have to state that neither probabilistic semi-backward nor partially backward bisimulation overcomes the issue that backward minimization does not preserve several timed probabilistic properties. Note that we cannot introduce a forward variant for both the requirements on rates *and* interactive transitions, as this would result in the definition of probabilistic forward bisimulation (cf. Definition 3.3.1).

5. Conclusion

In this thesis we introduced and formally defined a *probabilistic backward bisimulation relation* for Interactive Markov Chains, which, to the best of our knowledge, is a research area that has not been investigated at present. Our definition is based on the approaches by Sproston and Donatelli [21] for CTMCs in a backward manner and Hermanns and Katoen [13] for IMCs in a forward fashion. It turned out that our variant of pbb results in a quotient system that is in general not bisimilar to the original one but if the size of the equivalence classes, that are connected via Markovian transitions, are equal then we obtain a pbb-equivalent reduced system. This is due to the fact that in the quotient system, the Markovian transitions reflect an *average* value of moving in the original system from some state in one equivalence class to some state in another class. For each equivalence class C, pbb requires that each state within the same equivalence class is reachable via the same rate from class C. If the size of those classes linked by Markovian transitions coincide, the rates of these links in the quotient system agree with the rate by which every single state in an equivalence class can be reached by some state(s) in class C and hence, each state within the goal class will be pbb to the state representing the goal class in the quotient.

We have further proven that pbb is a congruence w.r.t. parallel composition, which is important if we want to minimize huge IMCs component-wise, i.e. computing the quotient for each component in the composed IMC, then applying parallel composition and obtaining a composed quotient system that is bisimilar to the original huge model. Clearly, this only holds if every original component is equivalent to its quotient and thus, as already discussed above, this will only be the case in the restrictive setting where the cardinalities of Markovian-connected classes are consistent.

To conclude chapter 3, we compared pbb to the probabilistic forward bisimulation invented by Hermanns and Katoen [13] with the finding that these two relations are incomparable.

Chapter 4 was dedicated to the quantitative timed analysis of IMCs and their quotients with respect to probabilistic backward bisimulation. Unfortunately, minimization under $\sim^b$ provides reduced models that do not preserve timed properties such as timed reachability, expected time, and long-run average. For that, we gave counterexamples where we computed the values for both original and quotient system, using the least fixed point computations provided by Zhang and Neuhäußer [23] and Guck et al. [11] for timed reachability and expected time, and an algorithm for the long-run average analysis introduced by Guck et al. [11]. However, for CTMCs the situation is different as already stated by Sproston and Donatelli [21]. The quotient CTMC preserves the average value of the maximum/minimum timed reachability probability. For IMCs, the problem roots in the permission of nondeterminism. If, for example, there exist two bisimilar states that are interactive, quotienting will merge them into a single state in the reduced system with probably two (or more) interactive transitions that lead to different equivalence classes. Recall that pbb does not take care of where interactive transitions move but only whether a state has an emanating interactive transition or not. Thus, an equivalence class might contain states that cannot reach a certain class in the original system. Then, when maxima or minima are computed, the corresponding algorithm (i.e. scheduler) chooses exactly *one* transition outgoing from that *class*, which will lead to the successor class from which the desired probability or amount of time will be maximal or, resp., minimal, *although* in the *original* system, this choice was made *independently for all states* contained in that equivalence class. This way information gets lost during pbb-minimization and the values of interest do not correspond in any way to that of the original system. Put in a nutshell, our definition of probabilistic backward bisimulation can reduce huge systems but the resulting model crucially fudges probabilistic timed properties.

Chapter 5 tried to overcome these problems by introducing two semi-backward variants of probabilistic backward bisimulation, first by transforming some conditions into forward variants. Our suggestion was that forward rules are applied to interactive states, whereas for Markovians backward conditions are kept. However, this does not affect the issue that original and reduced system are in general not bisimilar due to the calculation of the rates in the quotient. Applying a forward variant to the conditions imposed on rates, i.e. on Markovian states, results in a quotient that is bisimilar to the original system. Nevertheless, none of these approaches revealed satisfying results, as minimization still corrupts minimum and maximum timed reachability probabilities, expected times, and long-run averages.

For future work, we firstly need to find a compromise between these options, as the results of [21] and [20] together seem to be promising that the preservation of average probabilities and times might be possible (though algorithms generally have to be performed on the original system). And secondly, we need to adjust the conditions on Markovian states to generate a quotient that is bisimilar to the original IMC to enable a componentwise efficient analysis.

A. Appendix

A.1.

Lemma A.1.1 (PBB $\sim^b$ is an Equivalence)
For a fixed set of atomic propositions AP, the relation $\sim^b$ is an equivalence relation.

Proof Let AP be a set of atomic propositions. In the following, we show (1) reflexivity, (2) symmetry, and (3) transitivity of $\sim^b$.

(1) Note that for each state $s \in S$ of an IMC $\mathcal{I}$, it holds that $s \sim^b s$ and thus, the identity relation $\mathcal{R}_{id} = \{ (s,s) \mid s \in S \}$ is a probabilistic backward bisimulation for the pair $(\mathcal{I},\mathcal{I})$. It directly follows that, for any IMC $\mathcal{I}$, the relation $\sim^b$ on $(\mathcal{I},\mathcal{I})$ is reflexive.

(2) Let $\mathcal{R}$ be a pbb on $(\mathcal{I}_1,\mathcal{I}_2)$, where $\mathcal{I}_i$, $i = 1,2$, is an IMC. Let $\mathcal{R}^{-1}$ be defined as

$$\mathcal{R}^{-1} = \{ (s_2,s_1) \mid (s_1,s_2) \in \mathcal{R} \},$$

where the states of each pair in $\mathcal{R}$ are swapped. Then, $\mathcal{R}^{-1}$ satisfies conditions (A) and (B.1) of Definition 3.1.2. Swapping the ordering of states s_1 and s_2 in the given relation clearly does not affect the equality and equivalence properties required in conditions (B.2) - (B.5), and thus, we can directly conclude that $\mathcal{R}^{-1}$ is a pbb for $(\mathcal{I}_1,\mathcal{I}_2)$ and that the relation $\sim^b$ is symmetric.

(3) Let $\mathcal{I}_i = (S_i, Act, \longrightarrow_i, \Longrightarrow_i, s_{0,i}, AP, L_i)$, $i = 1,2,3$, be three IMCs. Assume $\mathcal{R}_{1,2}$ and $\mathcal{R}_{2,3}$ to be pbb's for the pairs $(\mathcal{I}_1,\mathcal{I}_2)$ and $(\mathcal{I}_2,\mathcal{I}_3)$, respectively, and define the relation $\mathcal{R}_{1,3} = \mathcal{R}_{1,2} \circ \mathcal{R}_{2,3}$ as follows:

$$\mathcal{R}_{1,3} = \{ (s_1,s_3) \mid \exists s_2 \in S_2.(s_1,s_2) \in \mathcal{R}_{1,2} \wedge (s_2,s_3) \in \mathcal{R}_{2,3} \}.$$

It suffices to show that each pair of states in $\mathcal{R}_{1,3}$ satisfies the conditions of Definition 3.1.2.

(A) Since $\mathcal{R}_{1,2}$ and $\mathcal{R}_{2,3}$ are pbb's, we have that $[s_{0,1}]_{\mathcal{R}_{1,2}} = [s_{0,2}]_{\mathcal{R}_{1,2}} = \{ s_{0,1}, s_{0,2} \}$ and $[s_{0,2}]_{\mathcal{R}_{2,3}} = [s_{0,3}]_{\mathcal{R}_{2,3}} = \{ s_{0,2}, s_{0,3} \}$. Hence, $[s_{0,1}]_{\mathcal{R}_{1,3}} = [s_{0,3}]_{\mathcal{R}_{1,3}} = \{ s_{0,1}, s_{0,3} \}$.

(B.1) Let $(s_1,s_3) \in \mathcal{R}_{1,3}$. By definition of $\mathcal{R}_{1,3}$, there exists a state $s_2 \in \mathcal{I}_2$, such that $(s_1,s_2) \in \mathcal{R}_{1,2}$ and $(s_2,s_3) \in \mathcal{R}_{2,3}$. It follows that $L_1(s_1) = L_2(s_2) = L_3(s_3)$.

(B.2) Let $(s_1,s_3) \in \mathcal{R}_{1,3}$ and assume there is a set $C_{1,3} \in (S_1 \uplus S_3)/\mathcal{R}_{1,3}$ and $\alpha \in Act$, such that $\mathbf{T}(C_{1,3},\alpha,s_1) \neq \mathbf{T}(C_{1,3},\alpha,s_3)$. W.l.o.g. assume that $\mathbf{T}(C_{1,3},\alpha,s_1) = 0$ and $\mathbf{T}(C_{1,3},\alpha,s_3) = 1$. By definition of $\mathbf{T}$, it follows that there exists at least one state $s \in C_{1,3}$ with $s \xrightarrow{\alpha} s_3$ and for all $s' \in C_{1,3}$, it holds that $s' \stackrel{\alpha}{\not\longrightarrow} s_1$. But then, since

$\mathcal{R}_{1,2}$ and $\mathcal{R}_{2,3}$ are pbb's and by definition of probabilistic backward bisimulation, there cannot exist a state $s_2 \in S_2$, such that $(s_1, s_2) \in \mathcal{R}_{1,2}$ and $(s_2, s_3) \in \mathcal{R}_{2,3}$ and consequently, the pair (s_1, s_3) of states cannot be included in the set $\mathcal{R}_{1,3}$. It follows that $\mathbf{T}(C_{1,3}, \alpha, s_1) = \mathbf{T}(C_{1,3}, \alpha, s_3)$ for all $C_{1,3} \in (S_1 \uplus S_3)/\mathcal{R}_{1,3}$, $(s_1, s_3) \in \mathcal{R}_{1,3}$ and $\alpha \in Act$.

(B.3) Let $(s_1, s_3) \in \mathcal{R}_{1,3}$ and $s_2 \in S_2$ be some state, such that $(s_1, s_2) \in \mathcal{R}_{1,2}$ and $(s_2, s_3) \in \mathcal{R}_{2,3}$. W.l.og. assume that $Post_\alpha(s_2) = \emptyset$. Since $\mathcal{R}_{1,2}$ and $\mathcal{R}_{2,3}$ are pbb's, it follows immediately that $Post_\alpha(s_1) = \emptyset = Post_\alpha(s_3)$. Hence, $Post_\alpha(s_1) = \emptyset$ if and only if $Post_\alpha(s_3) = \emptyset$ for all $(s_1, s_3) \in \mathcal{R}_{1,3}$.

(B.4) Let $C_{1,3}$ be an equivalence class in $(S_1 \uplus S_3)/\mathcal{R}_{1,3}$ with $Post_\tau(C_{1,3}) = \emptyset$, i.e. $C_{1,3} = \bigcup_{\substack{C_{1,2} \in (S_1 \uplus S_2)/\mathcal{R}_{1,2},\\ C_{2,3} \in (S_2 \uplus S_3)/\mathcal{R}_{2,3},\\ C_{1,2} \cap C_{2,3} \neq \emptyset}} ((C_{1,2} \cap S_1) \cup (C_{2,3} \cap S_3))$ is the set containing all states $s_1 \in S_1$ and $s_3 \in S_3$ that belong to an equivalence class $C_{1,2}$ under $\mathcal{R}_{1,2}$ and $C_{2,3}$ under $\mathcal{R}_{2,3}$, respectively, for which it holds that there exists at least one state $s_2 \in S_2$ that is inherited in both classes $C_{1,2}$ and $C_{2,3}$. Furthermore, let $(s_1, s_3) \in \mathcal{R}_{1,3}$. We have to prove that $\mathbf{R}(C_{1,3}, s_1) = \mathbf{R}(C_{1,3}, s_3)$, which can be seen as follows.

$$
\begin{aligned}
& \mathbf{R}(C_{1,3}, s_1) \\
& \text{(* by definition of } C_{1,3} \text{ and since } s_1 \text{ is not reachable from any class } C_{2,3} \text{ *)} \\
= \; & \sum_{\substack{C_{1,2} \in (S_1 \uplus S_2)/\mathcal{R}_{1,2}, \text{ where} \\ \exists C_{2,3} \in (S_2 \uplus C_3)/\mathcal{R}_{2,3}, s_2 \in S_2. \\ s_2 \in C_{1,2} \cap C_{2,3}}} \mathbf{R}(C_{1,2}, s_1) \\
& \text{(* since } \mathcal{R}_{1,2} \text{ is a pbb *)} \\
= \; & \sum_{\substack{C_{1,2} \in (S_1 \uplus S_2)/\mathcal{R}_{1,2}, \text{ where} \\ \exists C_{2,3} \in (S_2 \uplus C_3)/\mathcal{R}_{2,3}, s_2 \in S_2. \\ s_2 \in C_{1,2} \cap C_{2,3}}} \mathbf{R}(C_{1,2}, s_2) \\
& \text{(* by definition of } C_{1,3} \text{ *)} \\
= \; & \sum_{\substack{C_{2,3} \in (S_2 \uplus S_3)/\mathcal{R}_{2,3}, \text{ where} \\ \exists C_{1,2} \in (S_1 \uplus C_2)/\mathcal{R}_{1,2}, s_2 \in S_2. \\ s_2 \in C_{1,2} \cap C_{2,3}}} \mathbf{R}(C_{2,3}, s_2) \\
& \text{(* since } \mathcal{R}_{2,3} \text{ is a pbb *)} \\
= \; & \sum_{\substack{C_{2,3} \in (S_2 \uplus S_3)/\mathcal{R}_{2,3}, \text{ where} \\ \exists C_{1,2} \in (S_1 \uplus C_2)/\mathcal{R}_{1,2}, s_2 \in S_2. \\ s_2 \in C_{1,2} \cap C_{2,3}}} \mathbf{R}(C_{2,3}, s_3) \\
& \text{(* by definition of } C_{1,3} \text{ and since } s_3 \text{ is not reachable from any class } C_{1,2} \text{ *)} \\
= \; & \mathbf{R}(C_{1,3}, s_3)
\end{aligned}
$$

(B.5) Let (s_1, s_3) be a pair in $\mathcal{R}_{1,3}$ and assume (w.l.o.g.) that $Post_\tau(s_1) = \emptyset$. By definition, there exists a state $s_2 \in S_2$, such that $(s_1, s_2) \in \mathcal{R}_{1,2}$ and $(s_2, s_3) \in \mathcal{R}_{2,3}$. Since $\mathcal{R}_{1,2}$ and $\mathcal{R}_{2,3}$ are pbb's, we have that $\mathbf{E}(s_1) = \mathbf{E}(s_2)$ and $\mathbf{E}(s_2) = \mathbf{E}(s_3)$ and hence, $\mathbf{E}(s_1) = \mathbf{E}(s_3)$.

It follows that the relation $\sim^b$ is transitive. ■

A.2.

	Original IMC $\mathcal{I}$	Quotient IMC $\mathcal{I}/\sim^b$
1	*#INITIALS*	*#INITIALS*
2	*s*0	*c*0
3	*#GOALS*	*#GOALS*
4	*s*3	*c*2
5	*s*4	*#TRANSITIONS*
6	*#TRANSITIONS*	*c*0 *tau*
7	*s*0 *tau*1	$*$ *c*1 1.0
8	$*$ *s*1 1.0	*c*1 !
9	*s*0 *tau*2	$*$ *c*2 5.0
10	$*$ *s*2 1.0	$*$ *c*3 5.0
11	*s*1 !	*c*2 !
12	$*$ *s*3 5.0	$*$ *c*2 3.0
13	$*$ *s*4 5.0	*c*3 !
14	*s*2 !	$*$ *c*2 1.0
15	$*$ *s*5 5.0	
16	$*$ *s*6 5.0	
17	*s*3 !	
18	$*$ *s*4 3.0	
19	*s*4 !	
20	$*$ *s*3 3.0	
21	*s*5 !	
22	$*$ *s*3 1.0	
23	*s*6 !	
24	$*$ *s*4 1.0	

Table A.1.: .ma-files for Figure 4.1 for Timed Reachability Probability and Expected Time Analysis

Original IMC $\mathcal{I}$	Quotient IMC $\mathcal{I}/\sim^b$
#INITIALS	#INITIALS
s0	c0
#GOALS	#GOALS
s3	c2
s4	#TRANSITIONS
#TRANSITIONS	c0 tau
s0 tau1	* c1 1.0
* s1 1.0	c1 !
s0 tau2	* c2 5.0
* s2 1.0	* c3 5.0
s1 !	c2 !
* s3 5.0	* c4 3.0
* s4 5.0	c3 !
s2 !	* c2 1.0
* s5 5.0	c4 tau
* s6 5.0	* c0 1.0
s3 !	
* s7 3.0	
s4 !	
* s8 3.0	
s5 !	
* s3 1.0	
s6 !	
* s4 1.0	
s7 tau	
* s0 1.0	
s8 tau	
* s0 1.0	

Table A.2.: .ma-files for Figure 4.2 for Long-Run Average Analysis

	Original IMC $\mathcal{I}$	Quotient IMC $\mathcal{I}/{\sim^{sb}}$
1	*#INITIALS*	*#INITIALS*
2	*s*0	*c*0
3	*#GOALS*	*#GOALS*
4	*s*3	*c*2
5	*#TRANSITIONS*	*#TRANSITIONS*
6	*s*0 !	*c*0 !
7	∗ *s*1 1.0	∗ *c*1 6.0
8	∗ *s*2 5.0	*c*1 *tau*1
9	*s*1 *tau*	∗ *c*2 1.0
10	∗ *s*3 1.0	*c*1 *tau*2
11	*s*2 *tau*	∗ *c*3 1.0
12	∗ *s*4 1.0	*c*2 !
13	*s*3 !	∗ *c*2 20.0
14	∗ *s*3 20.0	*c*3 !
15	*s*4 !	∗ *c*2 2.0
16	∗ *s*3 2.0	

Table A.3.: .ma-files for Figure 4.4 for Timed Reachability Probability and Expected Time Analysis

	Original IMC $\mathcal{I}$	Quotient IMC $\mathcal{I}/{\sim^{sb}}$
1	*#INITIALS*	*#INITIALS*
2	*s*0	*c*0
3	*#GOALS*	*#GOALS*
4	*s*3	*c*2
5	*#TRANSITIONS*	*#TRANSITIONS*
6	*s*0 !	*c*0 !
7	∗ *s*1 1.0	∗ *c*1 6.0
8	∗ *s*2 5.0	*c*1 *tau*1
9	*s*1 *tau*	∗ *c*2 1.0
10	∗ *s*3 1.0	*c*1 *tau*2
11	*s*2 *tau*	∗ *c*3 1.0
12	∗ *s*4 1.0	*c*2 !
13	*s*3 !	∗ *c*0 20.0
14	∗ *s*0 20.0	*c*3 !
15	*s*4 !	∗ *c*2 2.0
16	∗ *s*3 2.0	

Table A.4.: .ma-files for Figure 4.5 for Long-Run Average Analysis

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